AGGREGATION METHODS FOR NEARLY UNCOUPLED SYSTEMS

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OUTLINE

- Intuitive overview of the Simon-Ando theory
- Intuitive derivation and discussion of basic A/D
- Analysis of errors in the basic A/D process

HISTORY

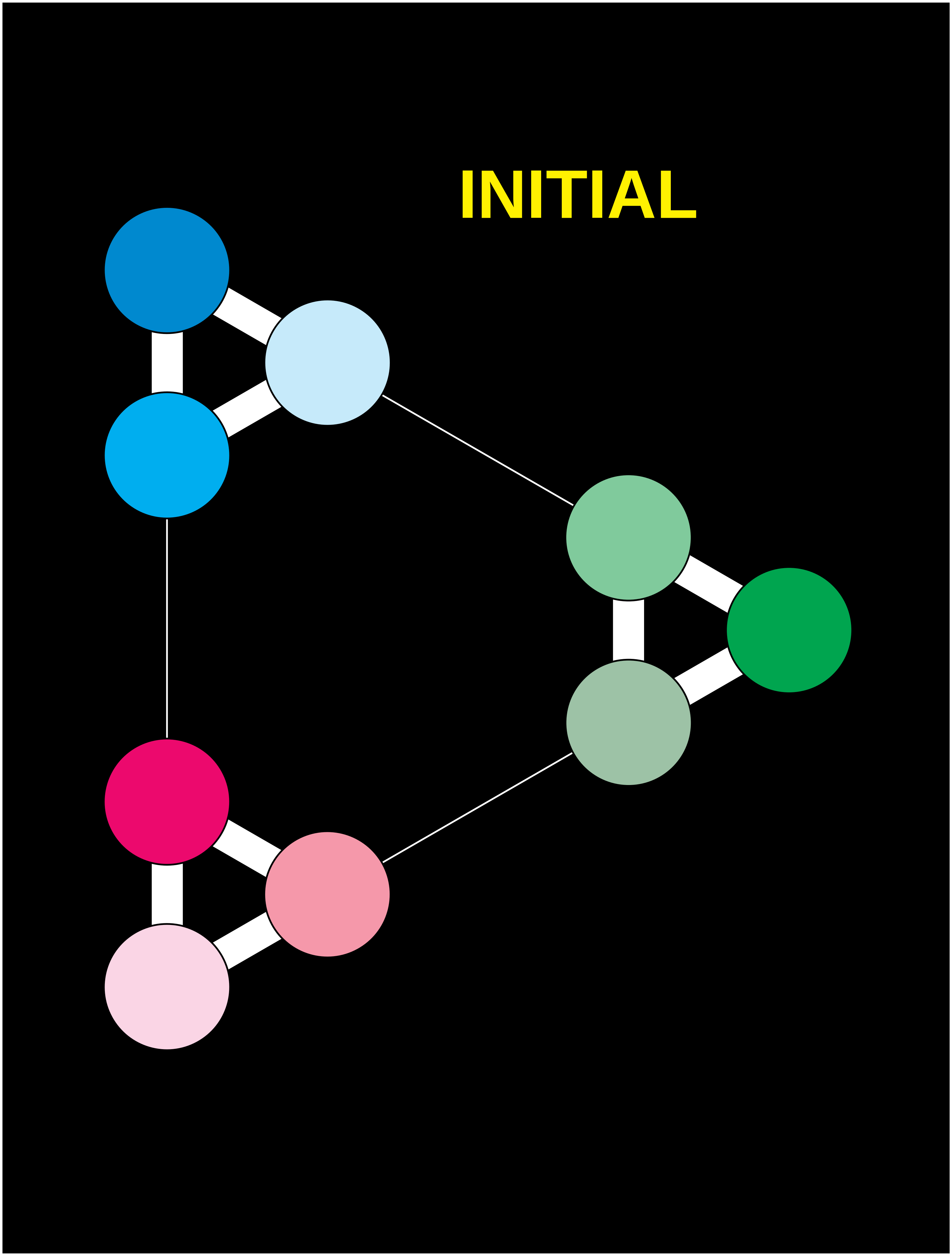
1961 H. Simon and A. Ando, ***Aggregation of variables in dynamic systems***, **Econometrica, Vol 29, pg. 111.**

1975 - 1977 **P. J. Courtois,**

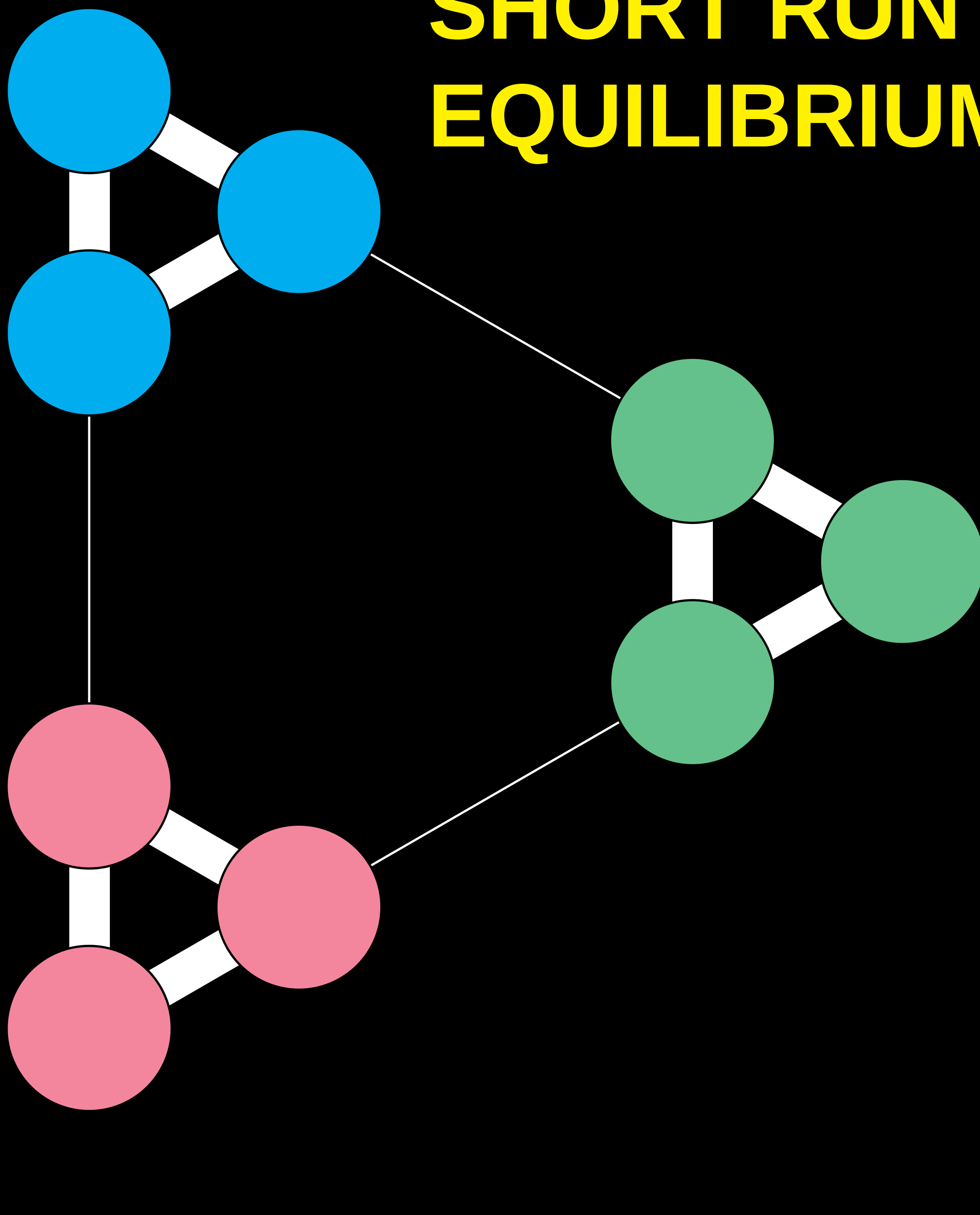
- **Econometrica, Vol 43, pg. 691.**
- ***Queueing And Computer System Applications*, Academic Press.**

1982 **F. Chatelin and W. Miranker,** ***Acceleration by aggregation of successive approximation methods*, LAA, Vol 43, pg.17.**

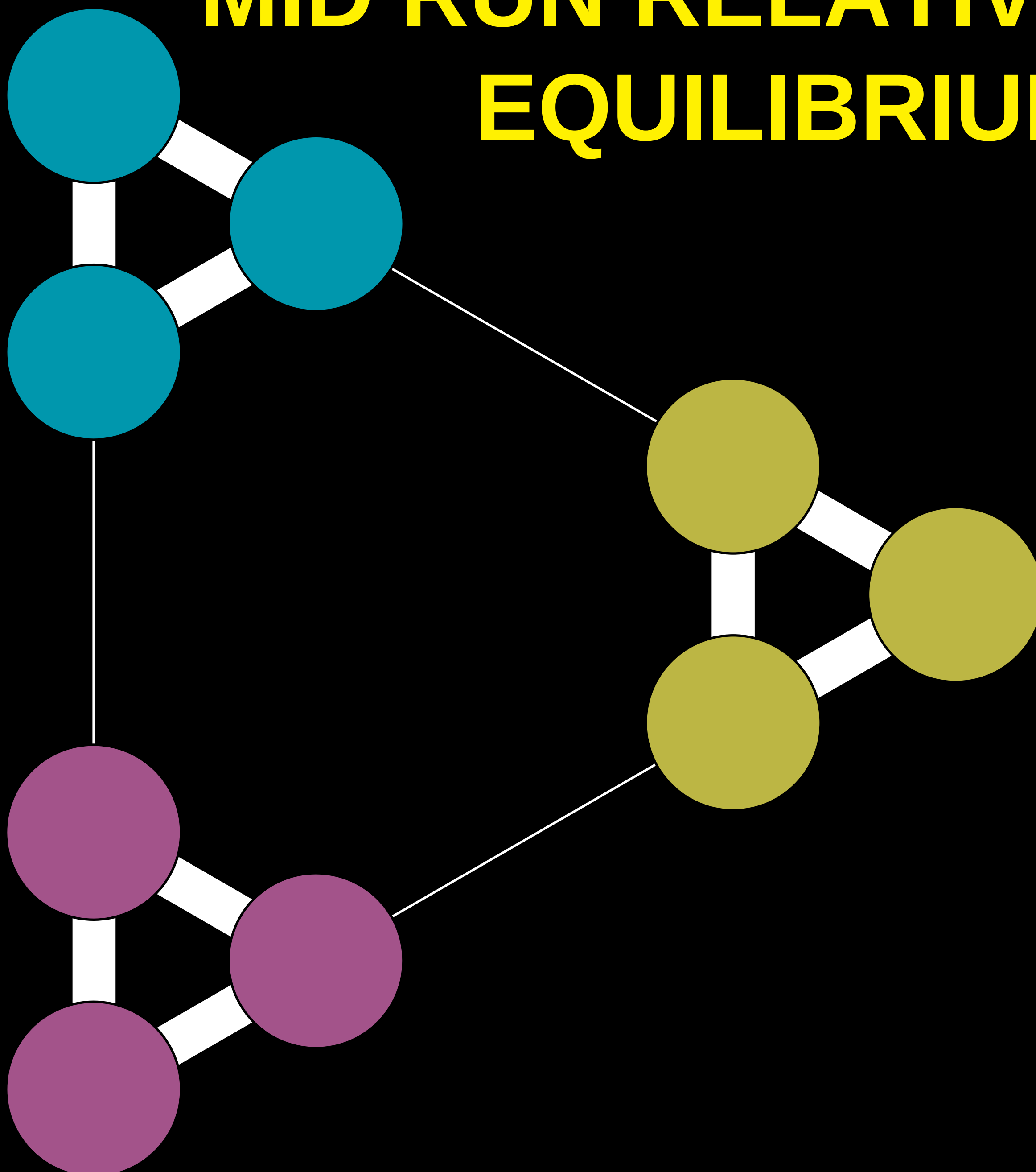
1984 **McAllister, G. Stewart, W. Stewart,** ***On a Rayleigh-Ritz refinement technique ...*** **LAA, Vol 60, pg.1.**


INITIAL

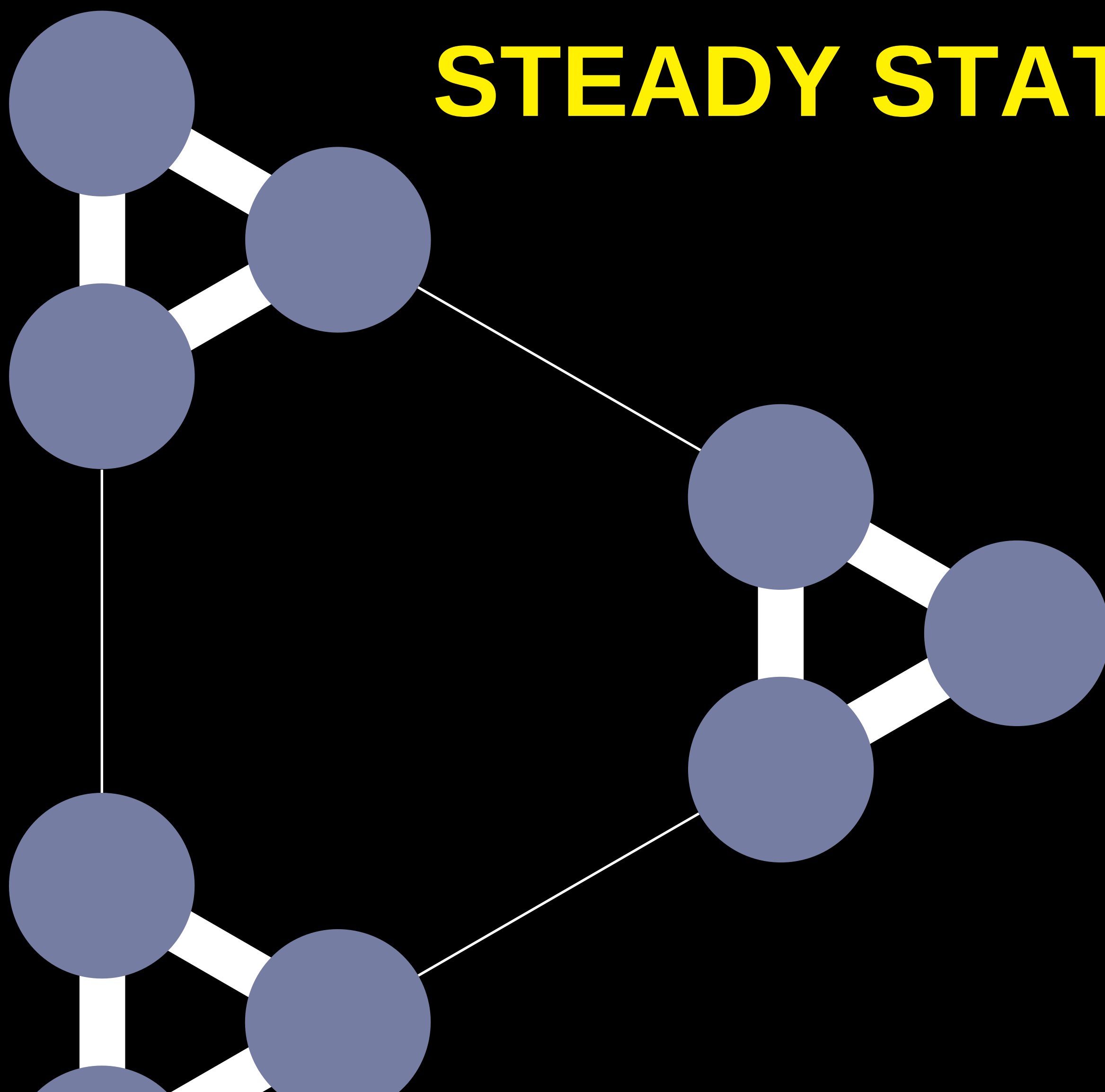
SHORT RUN EQUILIBRIUM



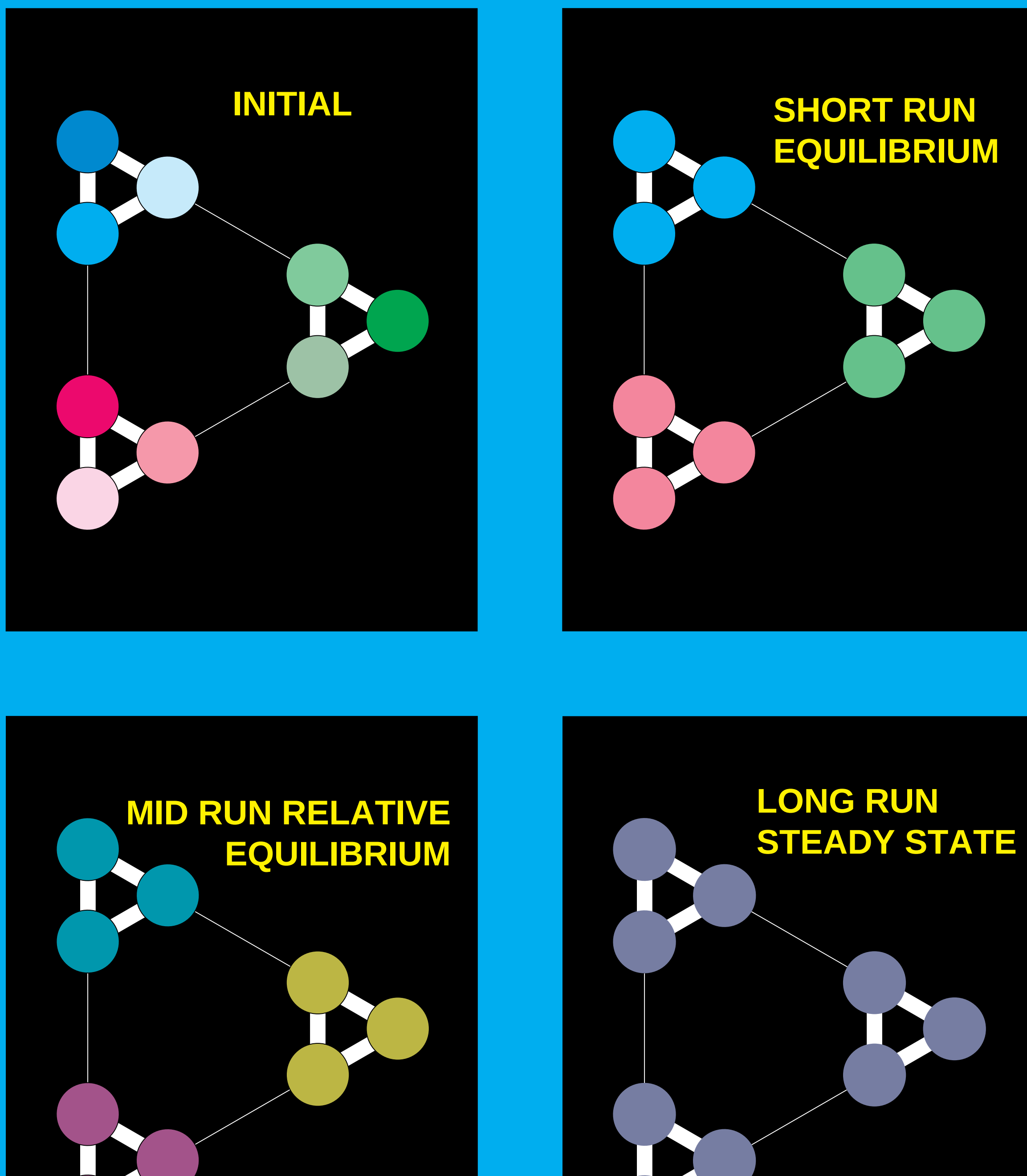
MID RUN RELATIVE EQUILIBRIUM



LONG RUN STEADY STATE



GOAL: Use short run info to estimate steady state



NEARLY UNCOUPLED MARKOV CHAINS

ASSUMPTIONS

- Large finite state space
- Homogeneous
- Irreducible
- Aperiodic
- Loosely connected clusters (say 3 of them) of closely connected states

NOTATION

Partitioned Transition Matrix

- $\mathbf{P} = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} & \mathbf{P}_{13} \\ \mathbf{P}_{21} & \mathbf{P}_{22} & \mathbf{P}_{23} \\ \mathbf{P}_{31} & \mathbf{P}_{32} & \mathbf{P}_{33} \end{bmatrix}$ $\quad\quad$ $\mathbf{e} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

$$\mathbf{P} \text{ ¸ } 0 \qquad\qquad \mathbf{Pe} = \mathbf{e}$$

Initial Distribution

- $¼^T(0) = \begin{bmatrix} ¼_1^T(0) & ¼_2^T(0) & ¼_3^T(0) \end{bmatrix}$

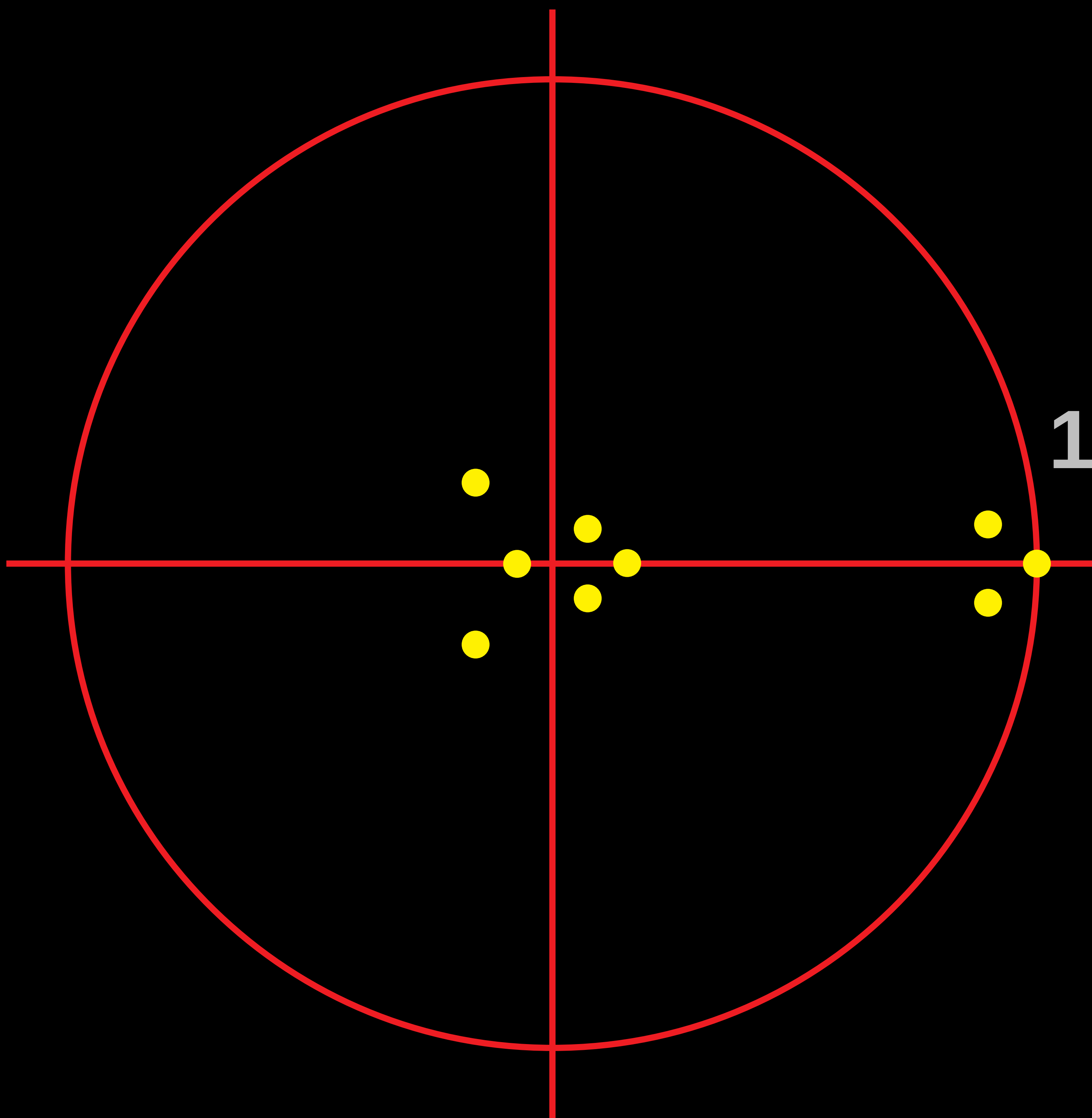
After t Steps

- $¼^T(t) = \begin{bmatrix} ¼_1^T(t) & ¼_2^T(t) & ¼_3^T(t) \end{bmatrix} = ¼_1^T(0)\mathbf{P}^t$

Steady State

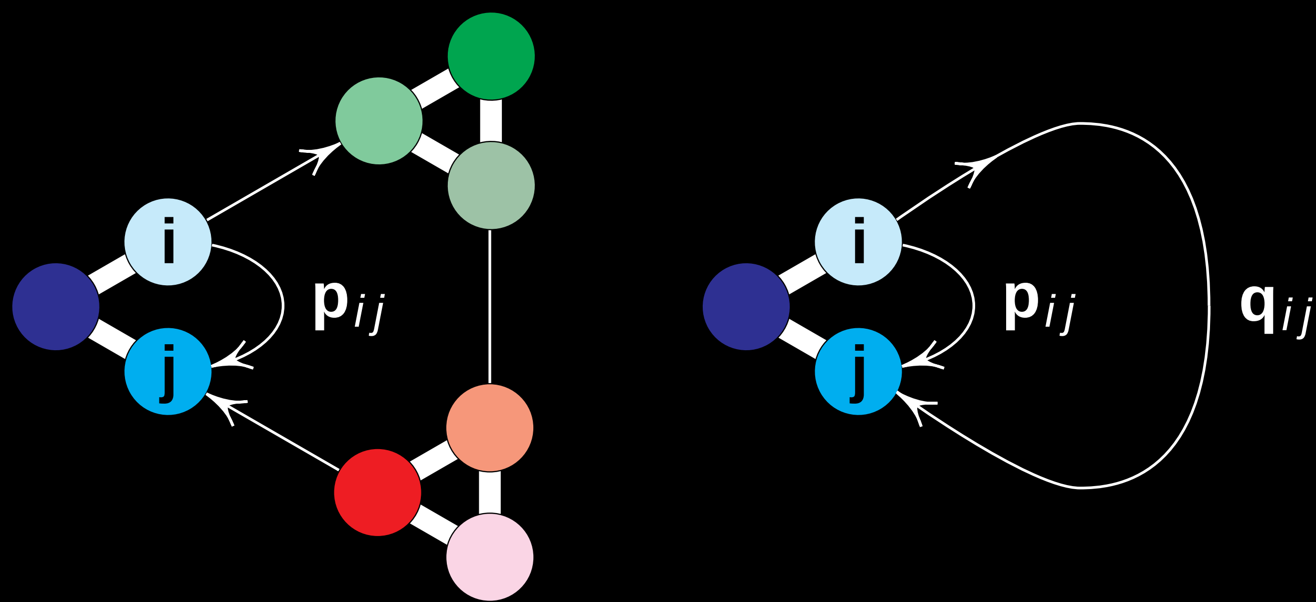
- $¼^T(t) \text{ ¡ ! } ¼^T = \begin{bmatrix} ¼_1^T & ¼_2^T & ¼_3^T \end{bmatrix}$
- $¼^T = ¼^T\mathbf{P}$

SPECTRUM OF P



CENSORED CHAIN

OBSERVE PROCESS IN ONLY ONE CLUSTER



- $\mathbf{p}_{ij}$ = **P**(i to j directly)
- $\mathbf{q}_{ij}$ = **P**(reenter cluster at j $^{\pm}$ leave from i)

CENSORED PROBABILITY

- $\mathbf{s}_{ij} = \mathbf{p}_{ij} + \mathbf{q}_{ij}$

CENSORED TRANSITION MATRIX

- $\mathbf{S} = \mathbf{P}_{11} + \mathbf{Q}_{11}$

QUESTION $\mathbf{Q}_{11}$=???

Q_{11} =???

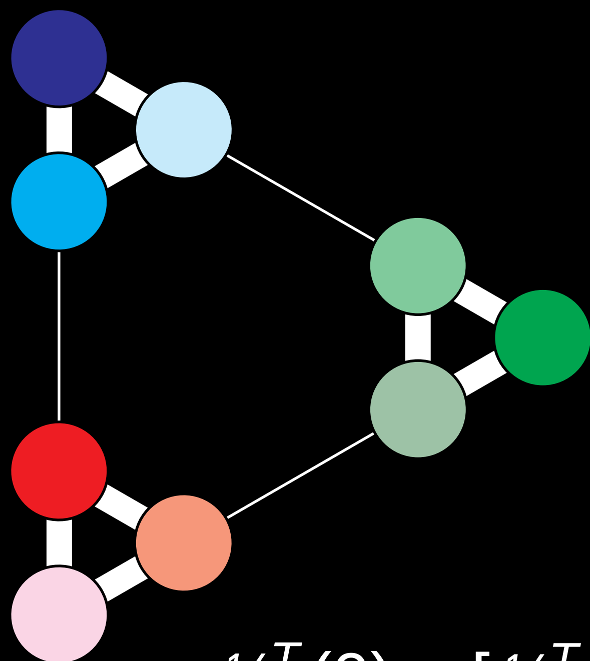
- $\mathbf{S}_{11} = \mathbf{P}_{11} + \mathbf{Q}_{11}$

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} & \mathbf{P}_{13} \\ \mathbf{P}_{21} & \mathbf{P}_{22} & \mathbf{P}_{23} \\ \mathbf{P}_{31} & \mathbf{P}_{32} & \mathbf{P}_{33} \end{bmatrix}$$

$$\mathbf{Q}_{11} = \begin{bmatrix} \mathbf{P}_{12} & \mathbf{P}_{13} \end{bmatrix} \begin{bmatrix} \mathbf{I} - \mathbf{P}_{22} & -\mathbf{P}_{23} \\ -\mathbf{P}_{32} & \mathbf{I} - \mathbf{P}_{33} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{P}_{21} \\ \mathbf{P}_{31} \end{bmatrix}$$

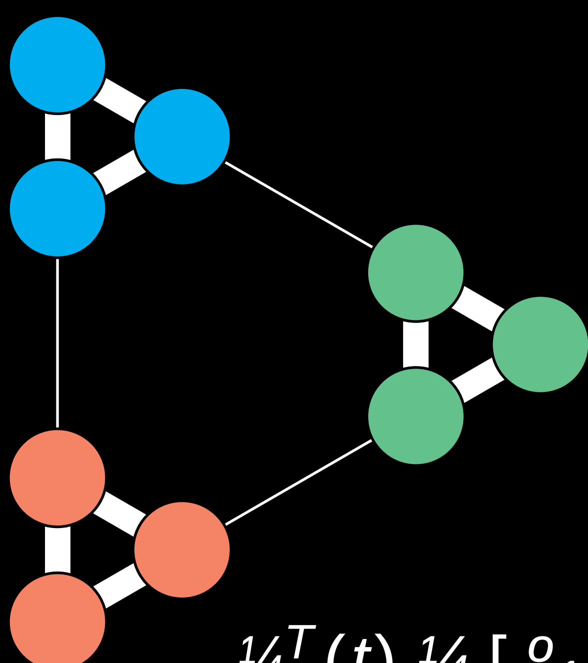
- $\mathbf{S}_{11} = \mathbf{P}_{11} + \mathbf{P}_{1*}(\mathbf{I} - \mathbf{P}_1)^{-1}\mathbf{P}_{*1}$
- **Similarly for $\mathbf{S}_{22}$ and $\mathbf{S}_{33}$**
- **$\mathbf{S}_{ii}$'s are called stochastic complements**
- $\mathbf{s}_i^T \mathbf{S}_{ii} = \mathbf{s}_i^T$
- $\mathbf{s}_i^T$ = steady state of i^{th} censored chain

BACK TO SIMON-ANDO



INITIAL

$$¼^{T}(0) = [\ ¼_1^{T}(0) \quad ¼_2^{T}(0) \quad ¼_3^{T}(0)\ ]$$

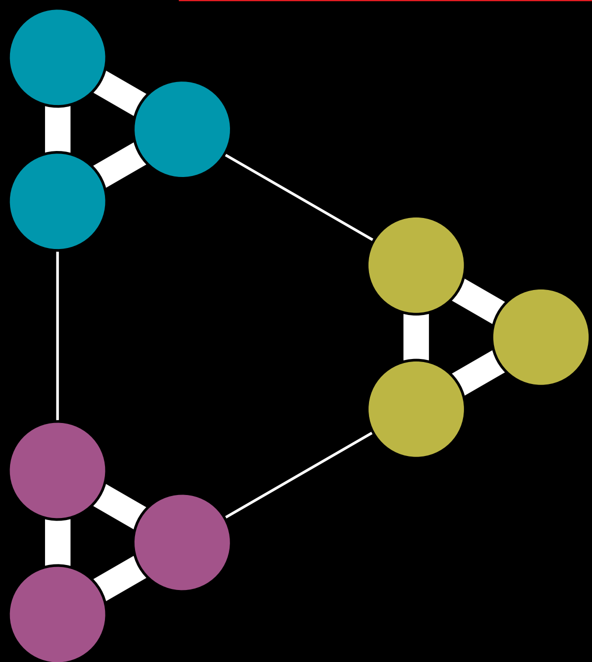


SHORT RUN
EQUILIBRIUM

$$¼^{T}(t)\ ¼\ [\ o_1 \mathbf{s}_1^{T} \quad o_2 \mathbf{s}_2^{T} \quad o_3 \mathbf{s}_3^{T}\ ]$$

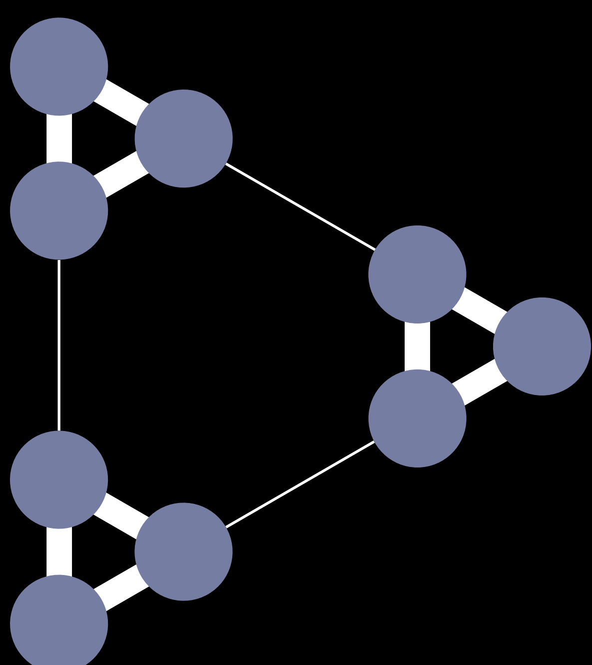
$$o_i = ¼_i^{T}(0)\mathbf{e} \quad \text{(constants)}$$

SIMON-ANDO (cont)



MID RUN RELATIVE EQUILIBRIUM

¼^{T}(t) ¼ [»$_1$(t)$\mathbf{s}_1^T$ »$_2$(t)$\mathbf{s}_2^T$ »$_3$(t)$\mathbf{s}_3^T$]



»$_i$(t) ! »$_i$

LONG RUN STEADY STATE

¼^{T}(t) ¡ ! ¼^{T} = [»$_1\mathbf{s}_1^T$ »$_2\mathbf{s}_2^T$ »$_3\mathbf{s}_3^T$]

THE PROBLEM

GOAL

- Estimate components in steady state

$$¼^T = [\,»_1 \mathbf{s}_1^T \quad »_2 \mathbf{s}_2^T \quad »_3 \mathbf{s}_3^T\,]$$

WE KNOW

- $\mathbf{s}_i^T$ can be estimated by observing the short-run equlibrium

REMAINING QUESTION

- How can we estimate the coupling factors $»_i$?

A/D OPERATORS

AGGREGATION (RESTRICTION)

- $\mathbf{A} : <^9 \; ! \; <^3$

DISAGGREGATION (PROLONGATION)

- $\mathbf{D} : <^3 \; ! \; <^9$
- $\mathbf{DA} = \mathbf{I}$

NATURAL CHOICES

- $$\mathbf{A}_{9 £ 3} = \begin{bmatrix} \mathbf{e} & 0 & 0 \\ 0 & \mathbf{e} & 0 \\ 0 & 0 & \mathbf{e} \end{bmatrix}$$

$$[\,\mathbf{p}_1^T \quad \mathbf{p}_2^T \quad \mathbf{p}_3^T\,]\,\mathbf{A} = [\,\mathbf{p}_1^T\mathbf{e} \quad \mathbf{p}_2^T\mathbf{e} \quad \mathbf{p}_3^T\mathbf{e}\,]$$

- $$\mathbf{D}_{3 £ 9} = \begin{bmatrix} \mathbf{s}_1^T & 0 & 0 \\ 0 & \mathbf{s}_2^T & 0 \\ 0 & 0 & \mathbf{s}_3^T \end{bmatrix}$$

$$[\,»_1 \quad »_2 \quad »_3\,]\,\mathbf{D} = [\,»_1\mathbf{s}_1^T \quad »_2\mathbf{s}_2^T \quad »_3\mathbf{s}_3^T\,]$$

COUPLING MATRIX

DEFINITION

- $\mathbf{C}$ ´ $(\mathbf{DPA})_{3£3} = £\, \mathbf{s}_i^T \mathbf{P}_{ij} \mathbf{e}\, ¤$

PROPERTIES

- $\mathbf{C}$ is irreducible and stochastic
- For $¼^T = [\, »_1 \mathbf{s}_1^T \quad »_2 \mathbf{s}_2^T \quad »_3 \mathbf{s}_3^T \,]$ and

  $»^T = [\, »_1 \quad »_2 \quad »_3 \,]$

  it must be the case that $»^T \mathbf{C} = »^T$

CONCLUSION

- If we can estimate the $\mathbf{s}_i^T$'s, then we can estimate the $»_i$'s by computing the steady state vector of **C**.

BASIC A/D ALGORITHM

1. Somehow compute an estimate $\hat{\mathbf{s}}_i^T$ of each $\mathbf{s}_i^T$

VARIOUS POSSIBILITIES

- Iterate until short-run approximate equilibrium is detected (until spectral components associated with small λ_i's are annihilated)

$$\pi^T(t) = [\,\mathbf{c}_1^T \quad \mathbf{c}_2^T \quad \mathbf{c}_3^T\,] \quad \Rightarrow \quad \hat{\mathbf{s}}_i^T = \frac{\mathbf{c}_i^T}{\mathbf{c}_i^T\mathbf{e}}$$

- Approximate each stochastic complement $\mathbf{S}_{ii} = \mathbf{P}_{ii} + \mathbf{P}_{i\star}(\mathbf{I} - \mathbf{P}_{\star})^{-1}\mathbf{P}_{\star i}$ with an irreducible $\hat{\mathbf{S}}_{ii} \geq 0$. Then compute

  $\hat{\mathbf{s}}_i^T$ = normalized (left) Perron vector of $\hat{\mathbf{S}}_{ii}$

Question: How can $\hat{\mathbf{S}}_{ii}$ be obtained?

Answer: Melt mass of $\mathbf{P}_{i\star}$ into $\mathbf{P}_{ii}$

- $\hat{\mathbf{S}}_{ii} = \mathbf{P}_{ii}$ + diagonal
- $\hat{\mathbf{S}}_{ii} = \mathbf{P}_{ii}$ + low rank update
- $\hat{\mathbf{S}}_{ii} = \mathbf{P}_{ii}$ (i.e., ignore $\mathbf{P}_{i\star}$)

A/D ALGORITHM (Cont)

2. Approximate the coupling matrix $\mathbf{C} = \left[\mathbf{s}_i^T \mathbf{P}_{ij} \mathbf{e}\right]$ by computing
   - $\hat{\mathbf{C}} = \left[\hat{\mathbf{s}}_i^T \mathbf{P}_{ij} \mathbf{e}\right]$

3. Estimate the coupling vector $\lambda^T = [\lambda_1 \quad \lambda_2 \quad \lambda_3]$ with the steady state vector of $\hat{\mathbf{C}}$
   - Solve $\hat{\lambda}^T = \hat{\lambda}^T \hat{\mathbf{C}}$ for $\hat{\lambda}^T = [\hat{\lambda}_1 \quad \hat{\lambda}_2 \quad \hat{\lambda}_3]$

4. Approximate $\pi^T = [\lambda_1 \mathbf{s}_1^T \quad \lambda_2 \mathbf{s}_2^T \quad \lambda_3 \mathbf{s}_3^T]$ with
   - $\hat{\pi}^T = [\hat{\lambda}_1 \hat{\mathbf{s}}_1^T \quad \hat{\lambda}_2 \hat{\mathbf{s}}_2^T \quad \hat{\lambda}_3 \hat{\mathbf{s}}_3^T]$

NOT TWICE

THE IDENTITY

- $$\mathbf{s}_i^T = \frac{\pi_i^T}{\pi_i^T \mathbf{e}}$$

SUGGESTS ITERATING

- by using $\tilde{\pi}^T = [\,\xi_1 \tilde{\mathbf{s}}_1^T \quad \xi_2 \tilde{\mathbf{s}}_2^T \quad \xi_3 \tilde{\mathbf{s}}_3^T\,]$ from 1^{st} A/D step to build 2^{nd} approximation of $\mathbf{s}_i^T$

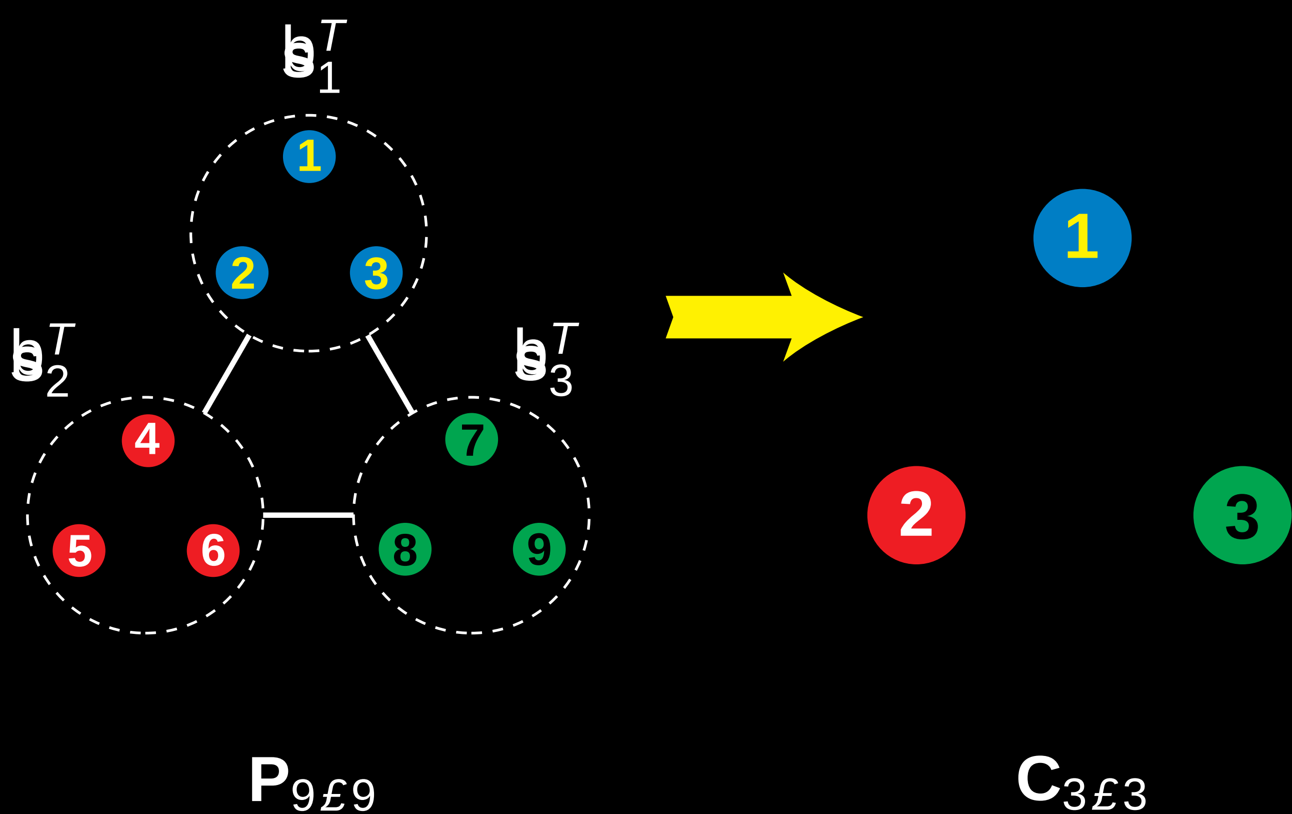
BUT

$$\hat{\mathbf{s}}_i^T = \frac{\xi_i \tilde{\mathbf{s}}_i^T}{\xi_i \tilde{\mathbf{s}}_i^T \mathbf{e}} = \frac{\xi_i \tilde{\mathbf{s}}_i^T}{\xi_i} = \tilde{\mathbf{s}}_i^T$$

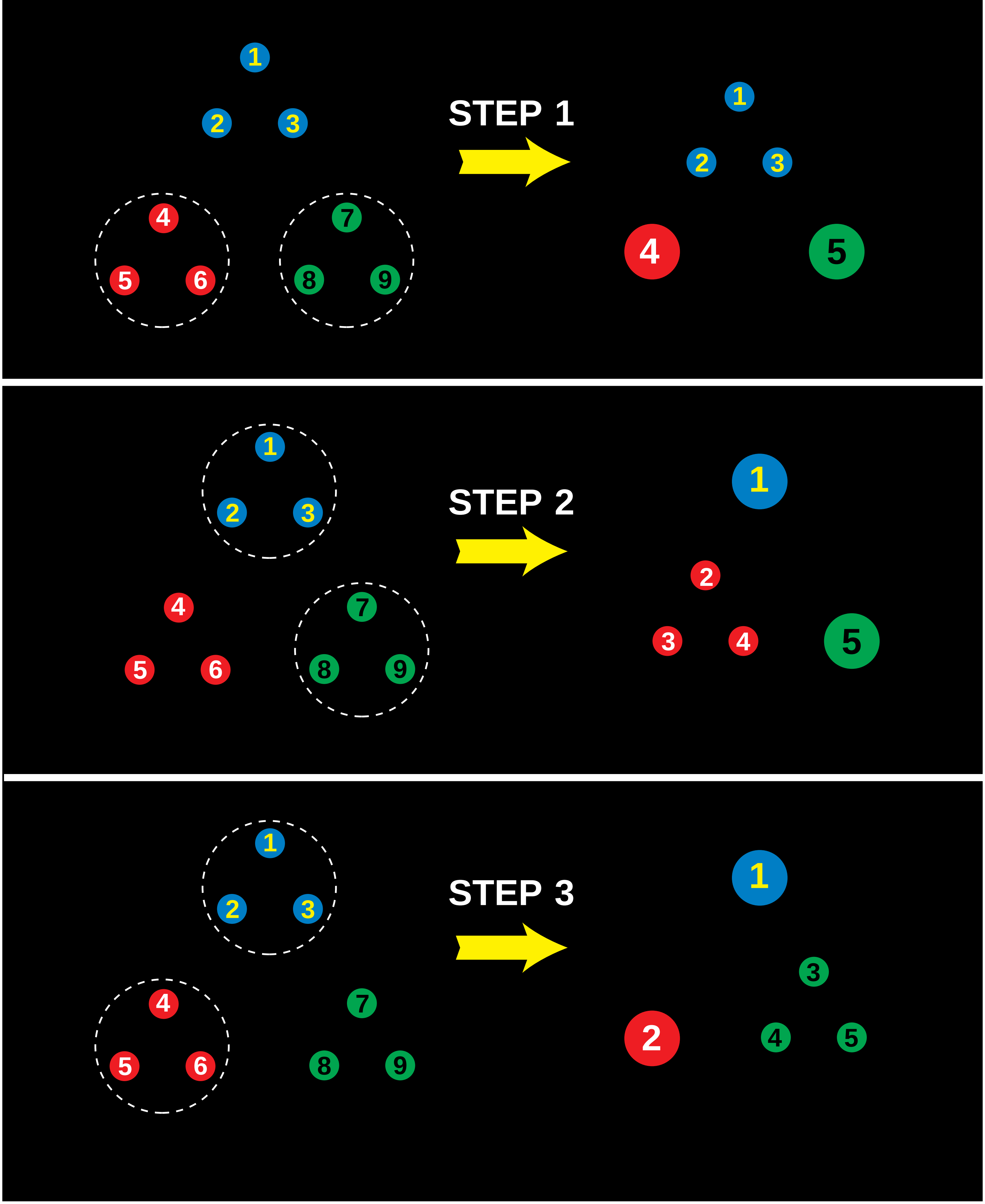
SO WE MUST ITERATE IN OTHER WAYS

- Restart iteration with $\pi^T(0) \leftarrow \tilde{\pi}^T$
  - A/D acts an accelerator to an underlying smoothing iteration (e.g., power method)
- Dynamic re-aggregation
  - Use an A/D output $\tilde{\pi}^T$ as input to different aggregation stratedgy

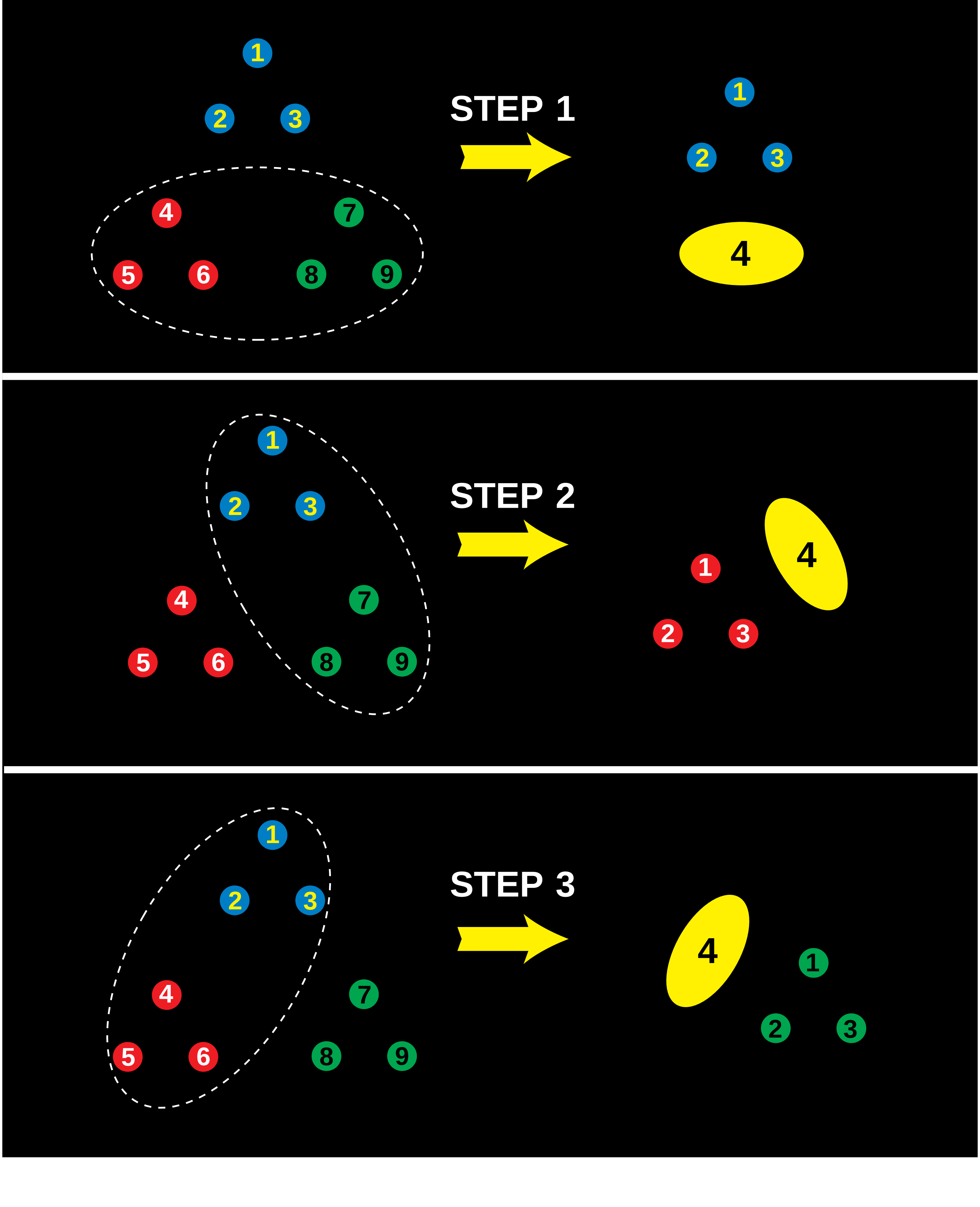
STANDARD A/D



DYNAMIC RE-AGGREGATION



DYNAMIC RE-AGGRE. (alt)



BASIC A/D ERRORS

INTERMEDIATE ERRORS

$$\hat{\mathbf{S}}_{ii} \approx \mathbf{S}_{ii} = \mathbf{P}_{ii} + \mathbf{P}_{i*}(\mathbf{I} - \mathbf{P}_i)^{-1}\mathbf{P}_{*i}$$

- $\Delta_i = \left\| \mathbf{S}_{ii} - \hat{\mathbf{S}}_{ii} \right\|_1 \qquad \Delta = \max_i \{\Delta_i\}$

$$\hat{\mathbf{s}}_i^T \hat{\mathbf{S}}_{ii} = \hat{\mathbf{s}}_i^T$$

- $\delta_i = \left\| \mathbf{s}_i^T - \hat{\mathbf{s}}_i^T \right\|_1 \qquad \delta = \max_i \{\delta_i\}$

$$\hat{\mathbf{C}} = \left[\hat{\mathbf{s}}_i^T \mathbf{P}_{ij} \mathbf{e}\right] \approx \mathbf{C} = \left[\mathbf{s}_i^T \mathbf{P}_{ij} \mathbf{e}\right]$$

- $\mathbf{F} = \mathbf{C} - \hat{\mathbf{C}}$

$$\hat{\lambda}^T \hat{\mathbf{C}} = \hat{\lambda}^T$$

- $\mathbf{f}^T = \lambda^T - \hat{\lambda}^T$

$$\hat{\pi}^T = [\hat{\lambda}_1 \hat{\mathbf{s}}_1^T \quad \hat{\lambda}_2 \hat{\mathbf{s}}_2^T \quad \hat{\lambda}_3 \hat{\mathbf{s}}_3^T]$$

THE BIG ONE

- $\left\| \pi^T - \hat{\pi}^T \right\|_1 \le ???$

BASIC A/D ERRORS

INTERMEDIATE ERRORS

$$\widehat{\mathbf{S}}_{ii} \approx \mathbf{S}_{ii} = \mathbf{P}_{ii} + \mathbf{P}_{i*}(\mathbf{I} - \mathbf{P}_i)^{-1}\mathbf{P}_{*i}$$

- $\Delta_i = \left\| \mathbf{S}_{ii} - \widehat{\mathbf{S}}_{ii} \right\|_\infty \qquad \Delta = \max_i \{\Delta_i\}$

$$\widehat{\mathbf{s}}_i^T \widehat{\mathbf{S}}_{ii} = \widehat{\mathbf{s}}_i^T$$ ← **Perturbed E-Vector Problem**

- $\delta_i = \left\| \mathbf{s}_i^T - \widehat{\mathbf{s}}_i^T \right\|_\infty \qquad \delta = \max_i \{\delta_i\}$

$$\widehat{\mathbf{C}} = \left[\widehat{\mathbf{s}}_i^T \mathbf{P}_{ij} \mathbf{e}\right] \approx \mathbf{C} = \left[\mathbf{s}_i^T \mathbf{P}_{ij} \mathbf{e}\right]$$

- $\mathbf{F} = \mathbf{C} - \widehat{\mathbf{C}}$

$$\widehat{\xi}^T \widehat{\mathbf{C}} = \widehat{\xi}^T$$ ← **Perturbed E-Vector Problem**

- $\mathbf{f}^T = \xi^T - \widehat{\xi}^T$

$$\widehat{\pi}^T = [\,\widehat{\xi}_1 \widehat{\mathbf{s}}_1^T \quad \widehat{\xi}_2 \widehat{\mathbf{s}}_2^T \quad \widehat{\xi}_3 \widehat{\mathbf{s}}_3^T\,]$$

THE BIG ONE

- $\left\| \pi^T - \widehat{\pi}^T \right\|_\infty \le$???

OLD ANALYSIS

MAXIMUM DEGREE OF COUPLING

- $² = 2 \max_i f k \mathbf{P}_{i¤} k_1 g$

COURTOIS (1977)

Theorem $\left\| ¼^T \; ¡ \; ½^T \right\|_1 = O(²)$

Proof Entire third chapter of his book

DIFFICULT POINTS

- Unknown Order Constants
- Asymptotic Arguments
- For validity, ² must be “small” but it can’t be determined how small “small” must be.
- Cumbersome Notation & Hard To Follow

DIFFERENT APPROACH

THE GROUP INVERSE

- $\mathbf{Q} = (\mathbf{I} - \mathbf{P}) \in$ multiplicative group

- $\mathbf{Q} = \mathbf{X}^{-1} \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{B} \end{bmatrix} \mathbf{X} \quad \Rightarrow \quad \mathbf{Q}^{\#} = \mathbf{X}^{-1} \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{B}^{-1} \end{bmatrix} \mathbf{X}$

- AKA Kato's reduced resolvent at $z = 0$

$$(z\mathbf{I} - \mathbf{Q})^{-1} = \frac{\mathbf{G}_{-1}}{z} + \mathbf{G}_0 + z\mathbf{G}_1 + z^2\mathbf{G}_2 + \cdots$$

$$\mathbf{G}_0 = \frac{1}{2\pi i} \oint_{C_0} \frac{(z\mathbf{I} - \mathbf{Q})^{-1}}{z} = \mathbf{Q}^{\#}$$

PERTURBED EIGENVECTORS

- Consider $\pi^T \mathbf{P} = \pi^T$ and $\tilde{\pi}^T \tilde{\mathbf{P}} = \tilde{\pi}^T$

- If $\mathbf{E} = \mathbf{P} - \tilde{\mathbf{P}}$ then $\pi^T - \tilde{\pi}^T = \tilde{\pi}^T \mathbf{E}\mathbf{Q}^{\#}$

ERGODICITY COEFFICIENTS

APPROXIMATE ¸ $_2$(**P**) WITH ¿ (**P**)

- ¿ (**P**$_1$**P**$_2$) · ¿ (**P**$_1$)¿ (**P**$_2$)
- 0 · ¿ (**P**) · 1
- ¿ (**P**) = 0 if and only if **P** = **e**¼^T

NORMS ALONE DON'T WORK WELL

- *k***P***k*$_1$ > 1 possibly
- *k***P***k*$_2$ ¸ 1 possibly
- *k***P***k* $_1$ = 1 for all **P**

A PARTICULAR ¿

$$¿(\mathbf{P}) = \max_{i;j} \frac{1}{2} \sum_{k=1}^{n} j p_{ik} \; ¡ \; p_{jk} j$$

PROPERTIES

- $¿(\mathbf{P}_1\mathbf{P}_2) \; · \; ¿(\mathbf{P}_1)¿(\mathbf{P}_2)$
- $0 \; · \; ¿(\mathbf{P}) \; · \; 1$ (= 1 iff 9 pair of $?$ rows)
- $¿(\mathbf{P}) = 0$ if and only if $\mathbf{P} = \mathbf{e}¼^T$
- $j ¸_2(\mathbf{P}) j \; · \; ¿(\mathbf{P})$ (Bauer, Deutch, Stoer, `69)
- $k\mathbf{FP}k_1 \; · \; k\mathbf{F}k_1 \, ¿(\mathbf{P})$ whenever $\mathbf{Fe} = 0$
- $k\mathbf{F}(\mathbf{I} \; ¡ \; \mathbf{P})^{\#} k_1 \; · \; \dfrac{k\mathbf{F}k_1}{1 \; ¡ \; ¿(\mathbf{P})}$
- $¿(\mathbf{I} \; ¡ \; \mathbf{P})^{\#} \; · \; \dfrac{n}{\min_{¸ \neq 1} j1 \; ¡ \; ¸ j}$ (Seneta, `93)

ANALYSIS

LET

- $\hat{\mathbf{S}}_{ii} = \mathbf{P}_{ii} + \mathbf{M}_i$ where $\|\mathbf{M}_i\|_1 = \|\mathbf{P}_{i*}\|_1$

SO THAT

- $\Delta_i \equiv \|\mathbf{S}_{ii} - \hat{\mathbf{S}}_{ii}\|_1 \le \epsilon = 2\max_i \|\mathbf{P}_{i*}\|_1$

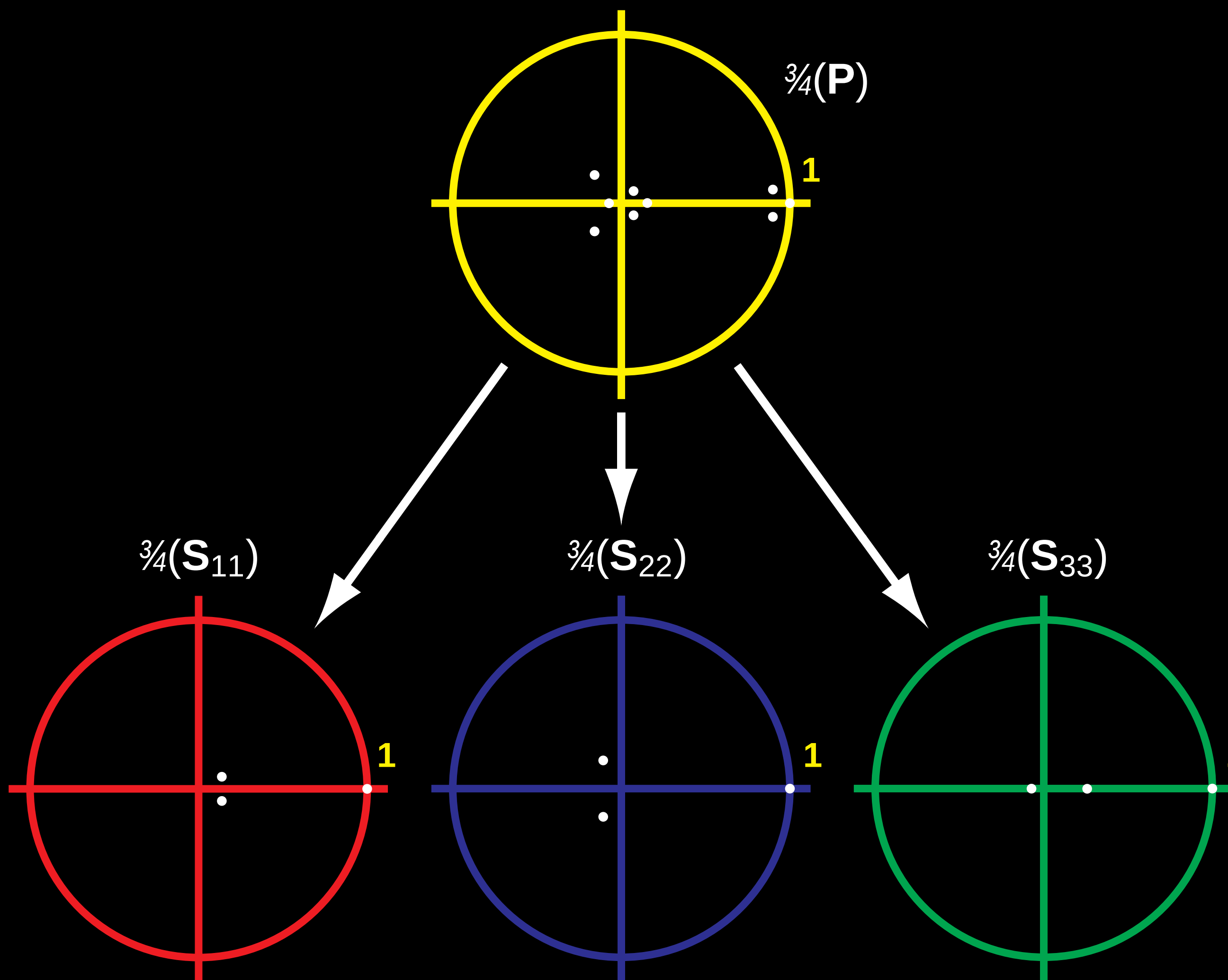
PROPOSITION

If $\hat{\mathbf{s}}_i^T \hat{\mathbf{S}}_{ii} = \hat{\mathbf{s}}_i^T$ and $\mathbf{S}_{ii}$ is $n_i \times n_i$

then

- $$\delta_i = \left\| \mathbf{s}_i^T - \hat{\mathbf{s}}_i^T \right\|_1 \le \Delta_i \, \kappa (\mathbf{I} - \mathbf{S}_{ii})^{\#} \le \frac{n_i \epsilon}{\min_{1 \ne \lambda \in \sigma(S_{ii})} |1 - \lambda|}$$

SPECTRUM SPLIT



- $\pm_i = \left\| \mathbf{s}_i^T \; i \; \hat{\mathbf{s}}_i^T \right\|_1 \cdot \dfrac{n_i{}^2}{\min\limits_{1 6 \, ¸ \, 2¾(S_{ii})} j1 \; i \; ¸ j} \; ¼ \; ^2 \, n_i$

- $\pm = \max\limits_i f \pm_i g \; \underset{»}{<} \; ^2 \max\limits_i n_i$

COUPLING ERROR

PROPOSITION For

$$\mathbf{F} = \mathbf{C} - \hat{\mathbf{C}} = \left[\mathbf{s}_i^T \mathbf{P}_{ij} \mathbf{e}\right] - \left[\hat{\mathbf{s}}_i^T \mathbf{P}_{ij} \mathbf{e}\right]$$

- $\|\mathbf{F}\|_\infty \leq \delta\epsilon \approx \epsilon^2 \max_i n_i$

PROPOSITION For $\hat{\xi}^T \hat{\mathbf{C}} = \hat{\xi}^T$

- $\|\xi^T - \hat{\xi}^T\|_1 \leq \|\mathbf{F}(\mathbf{I} - \mathbf{C})^\#\|_\infty \leq \dfrac{\|\mathbf{F}\|_\infty}{1 - \rho(\mathbf{C})}$

BUT

- $\mathbf{C} = \mathbf{DPA}$ and $\rho(\mathbf{D}) = \rho(\mathbf{A}) = 1$
- $\rho(\mathbf{C}) = \rho(\mathbf{DPA}) \leq \rho(\mathbf{D})\rho(\mathbf{P})\rho(\mathbf{A}) = \rho(\mathbf{P})$

THEREFORE

- $\|\xi^T - \hat{\xi}^T\|_1 \leq \dfrac{\delta\epsilon}{1 - \rho(\mathbf{P})} \approx \dfrac{\epsilon^2 \max_i n_i}{1 - \rho(\mathbf{P})}$

TOTAL ERROR

EXACT $\pi^T = [\xi_1 \mathbf{s}_1^T \quad \xi_2 \mathbf{s}_2^T \quad \xi_3 \mathbf{s}_3^T]$

A/D APPROX $\hat{\pi}^T = [\hat{\xi}_1 \hat{\mathbf{s}}_1^T \quad \hat{\xi}_2 \hat{\mathbf{s}}_2^T \quad \hat{\xi}_3 \hat{\mathbf{s}}_3^T]$

FOR 3 CLUSTERS

- $\left\| \pi^T - \hat{\pi}^T \right\|_1 \leq \dfrac{\delta^2}{1 - \tau(\mathbf{P})} + 3\delta$

FOR k CLUSTERS

- $\left\| \pi^T - \hat{\pi}^T \right\|_1 \leq \dfrac{\delta^2}{1 - \tau(\mathbf{P})} + k\delta$

$$\lesssim \frac{\epsilon^2 \max n_i}{1 - \tau(\mathbf{P})} + k\epsilon \max n_i$$

THIS IS A COMPUTABLE ESTIMATE

- All quantities in this estimate are directly available from **P**

SUMMARY

- The more uncoupled, the better A/D works
- Not good for closely coupled problems
- Can be iterated, but not in obvious ways
- Analysis is not easy for iterative cases