# The Life and Work of A. A. Markov

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on behalf of

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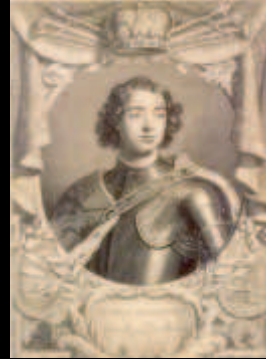North Carolina State University
Raleigh, NC

NSMC 9/5/2003

# Outline

- **Events before Markov**

- **Markov's Personal Life**

- **Markov's Academic Work**

# Russia in pre-Markov times

**Peter the First**



- The Great Reformer of Russia                    (in power from 1694–1725)

- forward-thinking, travelled West, visited other rulers, initiated Western reform in Russia

- absolute monarch: demanded subjects conform

- built St. Petersburg, the new capital, more accessible from West than Moscow

- member of Parisian Academy of Sciences

- Jan 24, 1724 edict established St. Petersburg Academy of Sciences with academy and gymnasium

(Russian Academy of Sciences)

# Academy of Sciences



- differed from European academies
  (not just a research body, but also an educational institution)

- Peter invited great foreign scientists to work for the Academy
  (Euler, N. and D. Bernoulli, C. Goldbach)

- Academic Positions at Academy

    Adjunct

    Extraordinary Academician

    Ordinary Academician

- Other Positions at Academy

    Corresponding Member

    Foreign Member

    Honorary Member – for royals and distinguished citizens

# Roots of Prob. theory begin at Academy

**Russian Firsts:**

- 1738 – first Russian paper on probability theory

  (D. Bernoulli, "Exposition of a new theory on the measurements of risk")

- 1846 – first Russian textbook on probability theory

  (V. Y. Bunyakovsky, "The Fundamentals of Mathematical Theory of Probability")

- 1846 – first Russian dissertation on probability theory

  (P. L. Chebyshev, "An Experience in an Elementary Analysis of the Probability Theory")

- 1840s–1860s – first Russian courses on probability theory taught at St. Petersburg University

  (Bunyakovsy and Chebyshev)

# Chebyshev's influence on Markov's work

- 1847 – Chebyshev obtains position at St. Petersburg University

- 1867 – "On Mean Values" generalized the law of large numbers

- 1887 – "On Two Theorems concerning Probability" generalized the CLT, presented method of moments

  (proof of CLT by method of moments later completed by Markov)

- Kolmogorov said that

  "Chebyshev was the first to estimate clearly and make use of such notions as random quantity and its expectation value."

# Chebyshev as a teacher



- Chebyshev had an impressive list of famous students.

  (Sokhotsky, Markov, Lyapunov, Voronoi)

- Chebyshev was renowned as a teacher. Lyapunov said,

  "His courses were not voluminous, and he did not consider the quantity of knowledge delivered; rather, he aspired to elucidate some of the most important aspects of the problems he spoke on. These were lively, absorbing lectures; curious remarks on the significance and importance of certain problems and scientific methods were always abundant. Sometimes he made a remark in passing, in connection with some concrete case they had considered, but those who attended always kept it in mind. Consequently, his lectures were highly stimulating; students received something new and essential at each lecture; he taught broader views and unusual standpoints."

# Markov's Academic Timeline

- born on June 14, 1856 in Ryazan, Russia

- early 1860s – Markovs move to St. Petersburg

- 1866 – A.A. attends gymnasium – poor at all subjects except mathematics

- "invents" new method for solution of linear differential equations. Informs Korkin and Zolotarev of his discovery, beginning lasting relationship with the St. Petersburg professors.

- 1874 – graduates from gymnasium, enters St. Petersburg University, takes courses from Chebyshev, Korkin, Zolotarev

- most active attendee of Korkin, Zolotarev's special home classes

- 1877 – receives gold medal for research, "On the Solution of Differential Equations with the help of Continuous Fractions"

- 1880 – Master's Thesis, "On the Binary Square Forms of the Positive Determinants"

# Markov's Academic Timeline (cont.)

- 1880 – Associate professor at St. Petersburg, teaches Analysis and Calculus. After Chebyshev retires, teaches Probability.

- 1880 – Doctoral Dissertation, "On Certain Applications of the Algebraic Continuous Fractions"

- 1886 – elected as adjunct to St. Petersburg Academy of Sciences, on Chebyshev's proposal. Extraordinary Professor at St. Petersburg University.

- 1889 – read paper "On a question by D. I. Mendeleev" at Academy session. Paper contains proof of Markov inequality for algebraic polynomials.

- 1890 – extraordinary academician of Academy of Sciences

- 1893 – ordinary professor at St. Petersburg University

- 1896 – ordinary academician of Academy of Sciences

- 1900 – published "On the Calculus of Probability" (4 Russian editions, 1 translated into German)

- 1905 – awarded honorary professorship at St. Petersburg University. Retires but continues to lecture on probability.

# ЗАПИСКИ

## ИМПЕРАТОРСКОЙ АКАДЕМІИ НАУКЪ.

## ТОМЪ ШЕСТЬДЕСЯТЪ ВТОРОЙ.

(съ 14 рисунками, 15 картами и 2 таблицами)

# ОБЪ ОДНОМЪ ВОПРОСѢ Д. И. МЕНДЕЛѢЕВА.

## А. Марковъ.

Въ настоящей статьѣ мы будемъ разсматривать совокупность тѣхъ цѣлыхъ функцій

$$f(z) = p_0 z^n + p_1 z^{n-1} + p_2 z^{n-2} + \ldots + p_{n-1} z + p_n,$$

степень которыхъ не превосходитъ даннаго цѣлаго числа $n$, а численныя значенія не превосходятъ другаго даннаго числа $L$ для всѣхъ значеній перемѣнной $z$, лежащихъ между данными предѣлами $a$ и $b > a$.

Итакъ

$$-L < f(z) < +L \quad \text{при} \quad a < z < b.$$

Спрашивается, какого предѣла не превосходитъ численное значеніе производной

$$f'(x) = n p_0 x^{n-1} + (n-1) p_1 x^{n-2} + \ldots + 2 p_{n-2} x + p_{n-1}$$

отъ $f(x)$ по $x$?

Такой вопросъ поставленъ Д. И. Менделѣевымъ, при $n = 2$, въ его сочиненіи «Изслѣдованіе водныхъ растворовъ по удѣльному вѣсу» (§ 86).

Отвѣтъ зависитъ отъ того насколько опредѣлено число $x$.

Мы различимъ два случая:

1) $x$ число данное,

2) $x$ произвольное число между $a$ и $b$.

Соотвѣтственно этому разсмотримъ двѣ задачи.

## Задача № 1.

Для даннаго числа $x$ найти наибольшее численное значеніе $f'(x)$.

### Рѣшеніе.

Обозначимъ черезъ $y$ ту изъ разсматриваемыхъ нами функцій $f(z)$, для которой $f'(x)$ численно достигаетъ наибольшаго значенія.

По условіямъ вопроса

$$-L \leqq y \leqq +L$$

для всѣхъ значеній $z$, лежащихъ между $a$ и $b$.

Изъ всѣхъ этихъ значеній $z$ обратимъ особое вниманіе на тѣ, при которыхъ $y$ равняется $\pm L$.

Пусть въ возрастающемъ порядкѣ они будутъ

$$\alpha_1, \alpha_2, \ldots, \alpha_i, \alpha_{i+1}, \ldots, \alpha_s.$$

Обозначивъ черезъ

$$y(\alpha_i)$$

значеніе $y$ при $z = \alpha_i$, равное $\pm L$, замѣтимъ, что рядъ $s-1$ отношеній

$$\frac{y(\alpha_2)}{y(\alpha_1)}, \frac{y(\alpha_3)}{y(\alpha_2)}, \ldots, \frac{y(\alpha_s)}{y(\alpha_{s-1})}$$

долженъ содержать по крайней мѣрѣ $n-1$ чиселъ равныхъ $-1$.

Дѣйствительно, въ противномъ случаѣ между цѣлыми функціями $n-2^{ой}$ степени отъ $z$ нетрудно найти безчисленное множество такихъ, отношенія которыхъ къ $y$ при

$$z = \alpha_1, \alpha_2, \ldots, \alpha_s$$

числа отрицательныя.

Если затѣмъ, умноживъ одну изъ нихъ

$$\varphi(z)$$

на $(z-x)^2$ и на достаточно малое положительное число $\varepsilon$, произведеніе

$$\varepsilon(z-x)^2 \varphi(z)$$

прибавимъ къ $y$, то получимъ цѣлую функцію

$$Y = y + \varepsilon(z-x)^2 \varphi(z)$$

$n^{ой}$ степени отъ $z$ и притомъ такую, что

при $a < z < b$   численное значеніе $Y < L$

и при $z = x$         $\dfrac{dY}{dz} = \dfrac{dy}{dz}$.

Наконецъ, если умножимъ $Y$ на отношеніе числа $L$ къ наибольшему численному значенію $Y$ при $a < z < b$, то полученная такимъ образомъ новая функція будетъ принадлежать къ числу разсматриваемыхъ нами функцій $f(z)$ и при $z = x$ ея производная численно больше $\dfrac{dy}{dz}$.

Итакъ $s$ не меньше $n$ и рядъ отношеній

$$\frac{y(\alpha_2)}{y(\alpha_1)}, \frac{y(\alpha_3)}{y(\alpha_2)}, \ldots, \frac{y(\alpha_s)}{y(\alpha_{s-1})} \tag{1}$$

содержитъ не менѣе $n-1$ чиселъ равныхъ $-1$.

Если $-1$ встрѣчается $n$ разъ въ ряду (1), то, какъ извѣстно, $y$ приводится къ

$$\pm L \cos n \arccos \frac{2z-a-b}{b-a} = \pm f_0(z).$$

Вмѣстѣ съ тѣмъ имѣемъ

$$\frac{dy}{dz} = \frac{\pm nL}{\sqrt{(z-a)(b-z)}} \sin n \arccos \frac{2z-a-b}{b-a} = \pm f_0'(z).$$

Изслѣдуемъ условія, при которыхъ наибольшее численное значеніе $f'(x)$ дѣйствительно равно численному значенію $f_0'(x)$.

Такъ какъ мы занимаемся численными значеніями, то изъ

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# WAHRSCHEINLICHKEITS-RECHNUNG

VON

## A. A. MARKOFF

NACH DER ZWEITEN AUFLAGE DES
RUSSISCHEN WERKES ÜBERSETZT

VON

### HEINRICH LIEBMANN

MIT 7 FIGUREN IM TEXT

LEIPZIG UND BERLIN

# Markov the Teacher

- interested in math education, protested vigorously against the "harmful experiments carried out in that field" at most schools.

- volunteers to lecture when the mathematics teacher as his son's college suddenly retires.

- emphasizes problem-solving.

- offers additional classes on Sundays and during vacation time.

- A. A. Markov, Jr. called his father's probability lectures "faultless".

- Mathematician Gunther remarked,

  "I know of instances where senior students ... would attend [his lectures] for the second time ... [even] after they had successfully passed Markov's course."

- Markov himself on teaching,

  "The alleged opinion that studies in seminars [in classes] are of the highest scientific nature, while exercises in solving problems are of the lowest [rank], is unfair. Mathematics to a considerable extent consists in solving problems, together with proper discussion can be of the highest scientific nature while studies in ... seminars might be of the lowest [rank]."

# Markov the Political Activist

- started young, his rebel tendencies appear in his school days

- refused to accept tsarist honors in protest against the exclusion of writer A. M. Gorky from the Academy

- 1902 – strongly disapproved of noblemen's honorary membership in Academy. Wrote a distasteful limerick about Duke Dundook's acceptance.

- maintained a very public animosity toward colleague Nekrasov, whose work he considered an "abuse of mathematics."

- 1904–1915 – wrote over 20 letters to newspapers about burning social and educational issues. Garnered himself the names, "Andrew the Furious" and the "Militant Academician".

- 1907 – renounced his membership in electorate when government dissolved the new Parliament.

- 1912 – wrote to Most Holy Synod asking to be excommunicated from Russian Orthodox Church.

    Synod replied, "[Markov] has seceded from God's Church and [we] expunged him from the lists of Orthodox believers."

# Markov the Political Activist

- Grodensky, biographer of Markov, recounts the following story from the last year of Markov's life, a time after one of the Russian revolutions.

On the 5th of March 1921 A. A. Markov communicated that on account of the absence of footwear he is not able to attend meetings of the Academy. A few weeks later the KUBU (Committee for Improvement of the Existence of Scientists), meeting under the chairmenship of A. M. Gorky, fulfilled the prosaic request of the famous mathematician. Time, however, provided a colorful sequel, of sorts, to this. At the meeting of the physico-mathematical section of the Academy of Science on the 25th May, Andrei Andreevich announced: "Finally, I received footwear; not only, however, is it stupidly stitched together, it does not in essence accord with my measurements. Thus, as before, I cannot attend meetings of the Academy. I propose placing the footwear received by me in the Ethnographic Museum as an example of the material culture of the current time, to which end I am ready to sacrifice it."

# Markov's Family

## Big Family

- Father (Andrei Grigorievich) married twice
  First Wife Nadezhda – 4 sons (Andrei A.), 2 daughters
  Second Wife Anna – 1 son (Vladimir), 2 daughters

## Brother Vladimir

- Andrei had much in common with younger half–brother (by 15 years)
- Vladimir studied mathematics at St. Petersburg
- 1892 – Vladimir extended his brother's inequality for polymonials
  to consider all derivatives
- 1897 – died at age 25 of tuberculosis
- Andrei completed and published Vladimir's unfinished master's
  thesis in 1897

## Young Andrei

- used crutches due to severe inborn deformity of knee
- has a knee operation at age 10 and afterwards walks with
  just a slight limp

# Markov's Personal Timeline

- 1856 – born on June 14 in Ryazan, Russia

- 1883 – marries Maria Valvatieva, whom he had tutored in math

- considered to be one of the best chess players in St. Petersburg

- 1903 – son, Andrei Andreevich Markov, Junior, born

  (renowned mathematician in fields of logic and algebra)

- 1922 – dies on July 20 at age 66 of bacterial infection in the blood caused by the last of his many leg operations on an aneurysm in his leg.  Buried in Mytrophany Cemetry in St. Petersburg.

A. LORENS     ST PETERSBOURG

# Markov's Mathematical

# Work on Chains

# Prehistory

Urn problems (Laplace, D. Bernoulli, Ehrenfests, Laplace)

Brownian motion (Bohlmann)

Extinction of families

Random walks like gambler's ruin (Buniakovsky)

Stock exchange (Bachelier)

Card games, gas mixture models (Poincare)

- Apparently, unrecognized by Markov (Nov. 11, 1910 letter to Chuprov)

"I most humbly beg you to point out to me those articles of Bohlmann and Bachelier to which you refer. Up to now I had thought that I was the first to dwell on the application of the law of large numbers to dependent variables..."

- After reading these articles, finds his work more general and correct (Nov. 15, 1910 and Nov. 18, 1910 letters to Chuprov)

Nov. 15: "I, of course, have seen Bachelier's article but strongly dislike it. I do not attempt to judge its significance for statistics but with respect to mathematics, it has no importance in my opinion. In any case, it does not contain an extension of Bernoulli's theorem to dependent variables."

Nov. 18: "The cases I indicated are NOT included in Bohlmann's cases, but contain them as special cases. There is a huge difference. I am prepared to admit that Bohlmann gave an elegant special formula, but he did not point out even one new (after my article) case of the generalization of Bernoulli's theorem."

# The Chain Gang

## The Concept of Chains – developments of predecessors

**1738** Jacob Bernoulli, "Ars Conjectandi" WLLN for sequences of independent binary r.v. $X_1, X_2, \ldots, X_n, \ldots$

$$P(X_i = 1) = p \text{ and } P(X_i = 0) = q$$

**1837** Simeon Poisson, "Recherches sur la Probablilite des Jugements en Matiere Criminelle et en Metiere Civile" generalized Bernoulli's theorem to include

$$P(X_i = 1) = p_i \text{ and } P(X_i = 0) = q_i$$

**1867** Pafnuty Chebyshev, "On Mean Values" generalized WLLN to any sequence of independent r.v. with bounded second moments

**1898** P. A. Nekrasov, "General properties of independent events in connection with the approximate computation of functions of very large numbers," erroneously claims that "independence is a necessary condition for the law of large numbers".

**1900** Andrei A. Markov, "Ischislenie veroyatnostej" further generalized WLLN to case when variances do not exist but all moments are bounded

# No. 3

6 November 1910

My dear Alexander Alexandrovich:

Of course I was also surprised by your reference to Bruns whom I consider a negligible quantity.

I can judge all work only from a strictly mathematical point of view and from this viewpoint it is clear to me that neither Bruns nor Nekrasov nor Pearson[1] has done anything worthy of note. You speak about some kinds of most general constructions, but I cannot find these constructions in their work.

Meanwhile I do find highly general theorems from authors whom you have entirely forgotten: A. M. Liapunov[2] and A. A. Markov. The unique service of P. A. Nekrasov, in my opinion, is namely this: he brings out sharply his delusion, shared, I believe, by many, that independence is a necessary condition for the law of large numbers. This circumstance prompted me to explain, in a series of articles, that the law of large numbers and Laplace's formula can apply also to dependent variables[3]. In this way a construction of a highly general character was actually arrived at, which P. A. Nekrasov can not even dream about.

I considered variables connected in a simple chain and from this came the idea of the possibility of extending the limit theorems of the calculus of probability also to a complex chain.

Independence is not required for the application of these theorems, but on the other hand it is necessary to assume existence of certain constant quantities. This existence is already assumed by the theory and therefore it is impossible to deduce this from the theory. And so I will stick to my opinion that your reference to Bruns and Nekrasov is wrong, as long as you do not cite for me their general constructions.

With complete respect,

A. MARKOV

# The Missing Link

- Markov wanted to relax the independent r.v. assumption

- began studying certain types of dependent r.v.

- considered sequences of dependent r.v. whose dependence trails off as the sequence extends.

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### and the CHAIN appeared!

BORN-ON DATE: 1906, "Extension of the law of large numbers to dependent variables"

Term MARKOV CHAIN debuts: 1926, Bernstein

# Markov's Definitions for Chains

## Simple Chain

"an infinite sequence $X_1, X_2, \ldots, X_k, X_{k+1}, \ldots$ of variables connected in such a way that $X_{k+1}$ for any $k$, is independent of $X_1, X_2, \ldots, X_{k-1}$ when $X_k$ is known." <span style="color:red">(1906, 1907 papers)</span>

## Homogeneous Chain

if conditional distributions of $X_{k+1}$ given $X_k$ are independent of time $k$.

## Complex Chain

"if every number is directly connected, not with a single, but with several preceding numbers".

# Markov's Theorems for Chains

## 1906

considered MCs with $\mathbf{P} > 0$ and proved ergodic theorem of MCs

<span style="color:red">(Independent of the initial distribution, chain converges to limiting, stationary distribution.)</span>

For ergodic chain, $\rho(\mathbf{P}) = 1$, all other $\lambda_i < 1$.

## 1908

dropped $\mathbf{P} > 0$ and begins to clarify what we call irreducible chains

"we consider only those chains $X_1, X_2, \ldots, X_n, \ldots$ where [the appearance of] some of the numbers...does not finally rule out the possibility of appearing others."

<span style="color:red">(all states are reachable from all others)</span>

His condition for irreducibility: "the determinant [of P] cannot be transformed into the product of several determinants of the same type."

<span style="color:red">(no symmetric permutation can transform P into block form, $\begin{bmatrix} X & Y \\ \mathbf{0} & Z \end{bmatrix}$)</span>

# Chains, WLLN, and the CLT

**WLLN (1906)**

- Markov used a 2-state MC (0 and 1) and Chebyshev's inequality to prove that the WLLN applies to his r.v. with Markov dependence.

- concluded that "the independence of variables does not produce the necessary condition for the validity of the law of large numbers."

**CLT (1907-1911)**

- used the method of moments from Chebyshev to prove that the CLT can be applied to his chains.

# Markov's Extensions to his Chains

**1910**  studies simple 2-state nonhomogeneous chain, "Investigation of the general case of trials connected in a chain"

**1911**  studies complex homogeneous chains with dependence on preceding two states, can't establish ergodic property

**1913**  publishes $3^{rd}$ edition of his book, "The Calculus of Probabilities"
  – adds work from 1906-1913 on chains
  – "Let us finish the article and the whole book with a good example of the dependent trials, which approximately can be regarded as a simple chain."

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The Famous First Application of Markov Chains

to

A. S. Pushkin's poem "Eugeny Onegin"

# A.S.Pushkin. Eugeny Onegin (1-3 chapter)

------------------------------------------------------------------
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## * CHAPTER I

I

My uncle was a man of virture,

When he became quite old and sick,

He sought respect and tried to teach me,

His only heir, verte and weak.

He had the fun, I had the sore,

But grecious goodness! what a bore!

To sit by bedplace day and night,

Not doing even step aside,

And what a cheep and cunning thing

To entertain the sad,

To serve around, make his bed,

To fetch the pills, to mourn and grim,

To sigh outloud, think along:

`God damn old man, why ain't you gone?'

Мой дядя самых честных правил,

Когда не в шутку занемог,

Он уважать себя заставил

И лучше выдумать не мог.

Его пример другим наука;

Но, боже мой, какая скука

С больным сидеть и днь и ночь

Не отходя ни шагу прочь!

Какое низкое коварство

Полуживого забавлять,

Ему подушки поправлять,

Печально подносить лекарство,

Вздыхать и думать про себя:

Когда же черт возьмет тебя!

II

So thought a playboy, young and funny,

While riding through the dust of road,

The only heir to the money,

That got his folks with help of Lord.

Так думал молодой повеса,

Летя в пыли на почтовых,

Всевышней волею Зевеса

Наследник всех своих родных.

Друзья Людмилы и Руслана!

С героем моего романа

Original: (english translation, all special characters except space removed, converted to small case):

my uncle a most worthy gentleman
when he fell seriously ill
constrained everyone to respect him
couldnt have done better if he tried
his behaviour was a lesson to us all
but god above what crushing boredom
to sit with the malingerer night and day
not moving even one footstep away
what demeaning hypocrisy
to amuse the halfdead codger
to fluff up his pillows and then
mournfully to bring him his medicine
to think to oneself and to sigh
when the devil will the old rascal die
so thought our young neerdowell
flying through the dust on a mail coach
by the supreme will of zeus
he was the inheritor of all his kin
good friends of ludmilla and ruslan
with the hero of my romance
allow me to make you acquainted ...



Top left corner of doublet count matrix:

|   | a | b | c | d | e | f | . | . |
|---|---|---|---|---|---|---|---|---|
| a | 1 | 23 | 26 | 58 | 4 | 2 | | |
| b | 16 | 1 | 0 | 0 | 51 | 0 | | |
| c | 31 | 0 | 4 | 0 | 73 | 0 | | |
| d | 36 | 0 | 2 | 7 | 76 | 0 | | |
| e | 123 | 2 | 23 | 118 | 63 | 14 | | |
| f | 31 | 0 | 0 | 2 | 38 | 17 | | |
| . | | | | | | | | |
| . | | | | | | | | |



Top left corner of stochastic matrix:

|   | a | b | c | d | e | f | . | . |
|---|---|---|---|---|---|---|---|---|
| a | 0.001 | 0.021 | 0.023 | 0.052 | 0.004 | 0.002 | | |
| b | 0.081 | 0.005 | 0.000 | 0.000 | 0.259 | 0.000 | | |
| c | 0.092 | 0.000 | 0.012 | 0.000 | 0.217 | 0.000 | | |
| d | 0.056 | 0.000 | 0.003 | 0.011 | 0.119 | 0.000 | | |
| e | 0.070 | 0.001 | 0.013 | 0.068 | 0.036 | 0.008 | | |
| f | 0.083 | 0.000 | 0.000 | 0.005 | 0.102 | 0.046 | | |
| . | | | | | | | | |
| . | | | | | | | | |



Random text generated by Markov chain:

abomy in tyoworeadshe ieablllothinear tim s wifon cow  ondowand abules ture in sethioriffour sshincore buspid
re tome ifound ry ien antha winglysth f nd the a ncik othted f fle ttoulead thapagousud conghinshe r che i arer
avelveg ornot tourig m dettonge ots bow ouco freed tyondevinglerolisps tul ad mo auspando mserin lisuriaset
lurisy wigld wichare toutais ddethent tlouteunde anonlend be t s hthules ad iem ded adch he tly hean hesms ita
oureliand te ppsushis atrues amby f hilell avecol t chasere alembow cotr itrever morericomanoffevetthin om ...

# Calculations for "Eugeny Onegin"

"The second question concerns an original statistical investigation which I have carried out and with which I propose to conclude my book. The character of the investigation, which embraces a sequence of 20,000 letters, is shown in the example below."

<span style="color:red">(Jan. 15, 1913 letter to Chuprov)</span>

stationary probability of vowel = .432

stationary probability of consonant = .568

probability of a vowel following a vowel = .128

probability of a vowel following a consonant = .663

## Besikovitch on Markov's book:

"great attention is paid to the simplest numerical examples which are discussed in unusual detail. And further, it is hardly possible to find a single mistake in these examples."

## Markov on calculating:

"Many mathematicians apparently believe that going beyond the field of abstract reasoning into the sphere of effective calculations would be humiliating."

## Table for Normal Distribution

produced a table with up to 11 digits. Table was used into the 1940s.

*Tabelle der Werte von* $\dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-t^2}\,dt.$

| $x$ | | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1,90 | 0,99 | 2790 | 2821 | 2851 | 2881 | 2912 | 2942 | 2972 | 3001 | 3031 | 3061 |
| 1,91 | 0,99 | 3090 | 3119 | 3148 | 3178 | 3207 | 3235 | 3264 | 3293 | 3321 | 3350 |
| 1,92 | 0,99 | 3378 | 3406 | 3435 | 3463 | 3490 | 3518 | 3546 | 3574 | 3601 | 3628 |
| 1,93 | 0,99 | 3656 | 3683 | 3710 | 3737 | 3764 | 3790 | 3817 | 3844 | 3870 | 3896 |
| 1,94 | 0,99 | 3923 | 3949 | 3975 | 4001 | 4026 | 4052 | 4078 | 4103 | 4129 | 4154 |
| 1,95 | 0,99 | 4179 | 4204 | 4229 | 4254 | 4279 | 4304 | 4329 | 4353 | 4378 | 4402 |
| 1,96 | 0,99 | 4426 | 4450 | 4475 | 4498 | 4522 | 4546 | 4570 | 4593 | 4617 | 4640 |
| 1,97 | 0,99 | 4664 | 4687 | 4710 | 4733 | 4756 | 4779 | 4802 | 4824 | 4847 | 4870 |
| 1,98 | 0,99 | 4892 | 4914 | 4937 | 4959 | 4981 | 5003 | 5025 | 5047 | 5068 | 5090 |
| 1,99 | 0,99 | 5111 | 5133 | 5154 | 5176 | 5197 | 5218 | 5239 | 5260 | 5281 | 5302 |
| | | | | | | | | | | | |
| 2,00 | 0,99 | 5322 | 5343 | 5363 | 5384 | 5404 | 5425 | 5445 | 5465 | 5485 | 5505 |
| 2,01 | 0,99 | 5525 | 5545 | 5564 | 5584 | 5604 | 5623 | 5643 | 5662 | 5681 | 5700 |
| 2,02 | 0,99 | 5719 | 5738 | 5757 | 5776 | 5795 | 5814 | 5832 | 5851 | 5870 | 5888 |
| 2,03 | 0,99 | 5906 | 5925 | 5943 | 5961 | 5979 | 5997 | 6015 | 6033 | 6050 | 6068 |
| 2,04 | 0,99 | 6086 | 6103 | 6121 | 6138 | 6156 | 6173 | 6190 | 6207 | 6224 | 6241 |
| 2,05 | 0,99 | 6258 | 6275 | 6292 | 6308 | 6325 | 6342 | 6358 | 6375 | 6391 | 6407 |
| 2,06 | 0,99 | 6423 | 6440 | 6456 | 6472 | 6488 | 6504 | 6519 | 6535 | 6551 | 6567 |
| 2,07 | 0,99 | 6582 | 6598 | 6613 | 6628 | 6644 | 6659 | 6674 | 6689 | 6704 | 6719 |
| 2,08 | 0,99 | 6734 | 6749 | 6764 | 6779 | 6794 | 6808 | 6823 | 6837 | 6852 | 6866 |
| 2,09 | 0,99 | 6880 | 6895 | 6909 | 6923 | 6937 | 6951 | 6965 | 6979 | 6993 | 7007 |
| | | | | | | | | | | | |
| 2,10 | 0,99 | 7021 | 7034 | 7048 | 7061 | 7075 | 7088 | 7102 | 7115 | 7128 | 7142 |
| 2.11 | 0,99 | 7155 | 7168 | 7181 | 7194 | 7207 | 7220 | 7233 | 7246 | 7258 | 7271 |
| 2,12 | 0,99 | 7284 | 7296 | 7309 | 7321 | 7334 | 7346 | 7358 | 7371 | 7383 | 7395 |
| 2,13 | 0,99 | 7407 | 7419 | 7431 | 7443 | 7455 | 7467 | 7479 | 7490 | 7502 | 7514 |
| 2,14 | 0,99 | 7525 | 7537 | 7548 | 7560 | 7571 | 7583 | 7594 | 7605 | 7616 | 7627 |
| 2,15 | 0,99 | 7639 | 7650 | 7661 | 7672 | 7683 | 7693 | 7704 | 7715 | 7726 | 7737 |
| 2,16 | 0,99 | 7747 | 7758 | 7768 | 7779 | 7789 | 7800 | 7810 | 7820 | 7831 | 7841 |
| 2,17 | 0,99 | 7851 | 7861 | 7871 | 7881 | 7891 | 7901 | 7911 | 7921 | 7931 | 7941 |
| 2,18 | 0,99 | 7951 | 7960 | 7970 | 7980 | 7989 | 7999 | 8008 | 8018 | 8027 | 8037 |
| 2,19 | 0,99 | 8046 | 8055 | 8065 | 8074 | 8083 | 8092 | 8101 | 8110 | 8119 | 8128 |
| | | | | | | | | | | | |
| 2,20 | 0,99 | 8137 | 8146 | 8155 | 8164 | 8173 | 8181 | 8190 | 8199 | 8207 | 8216 |
| 2,21 | 0,99 | 8224 | 8233 | 8241 | 8250 | 8258 | 8267 | 8275 | 8283 | 8292 | 8300 |
| 2,22 | 0,99 | 8308 | 8316 | 8324 | 8332 | 8340 | 8348 | 8356 | 8364 | 8372 | 8380 |
| 2,23 | 0,99 | 8388 | 8396 | 8403 | 8411 | 8419 | 8426 | 8434 | 8442 | 8449 | 8457 |
| 2,24 | 0,99 | 8464 | 8472 | 8479 | 8486 | 8494 | 8501 | 8508 | 8516 | 8523 | 8530 |
| 2,25 | 0,99 | 8537 | 8544 | 8552 | 8559 | 8566 | 8573 | 8580 | 8586 | 8593 | 8600 |
| 2,26 | 0,99 | 8607 | 8614 | 8621 | 8627 | 8634 | 8641 | 8648 | 8654 | 8661 | 8667 |
| 2,27 | 0,99 | 8674 | 8680 | 8687 | 8693 | 8700 | 8706 | 8712 | 8719 | 8725 | 8731 |
| 2,28 | 0,99 | 8738 | 8744 | 8750 | 8756 | 8762 | 8768 | 8775 | 8781 | 8787 | 8793 |
| 2,29 | 0,99 | 8799 | 8805 | 8810 | 8816 | 8822 | 8828 | 8834 | 8840 | 8845 | 8851 |
| | | | | | | | | | | | |
| 2,30 | 0,99 | 8857 | 8862 | 8868 | 8874 | 8879 | 8885 | 8890 | 8896 | 8902 | 8907 |
| 2,31 | 0,99 | 8912 | 8918 | 8923 | 8929 | 8934 | 8939 | 8945 | 8950 | 8955 | 8960 |
| 2,32 | 0,99 | 8966 | 8971 | 8976 | 8981 | 8986 | 8991 | 8996 | 9001 | 9006 | 9011 |
| 2,33 | 0,99 | 9016 | 9021 | 9026 | 9031 | 9036 | 9041 | 9045 | 9050 | 9055 | 9060 |
| 2,34 | 0,99 | 9065 | 9069 | 9074 | 9079 | 9083 | 9088 | 9093 | 9097 | 9102 | 9106 |
| 2,35 | 0,99 | 9111 | 9115 | 9120 | 9124 | 9129 | 9133 | 9137 | 9142 | 9146 | 9150 |
| 2,36 | 0,99 | 9155 | 9159 | 9163 | 9168 | 9172 | 9176 | 9180 | 9184 | 9189 | 9193 |
| 2,37 | 0,99 | 9197 | 9201 | 9205 | 9209 | 9213 | 9217 | 9221 | 9225 | 9229 | 9233 |
| 2,38 | 0,99 | 9237 | 9241 | 9245 | 9249 | 9252 | 9256 | 9260 | 9264 | 9268 | 9271 |
| 2,39 | 0,99 | 9275 | 9279 | 9282 | 9286 | 9290 | 9293 | 9297 | 9301 | 9304 | 9308 |

# A. A. Markov and A. A. Chuprov

- corresponded between 1910 and 1917, over 100 letters were found, Chuprov's letters from 1912, 1913, 1915 were lost

- both lived in St. Petersburg. Fortunate for us, they wrote rather than met, or phoned.

- wrote every 2-3 days in beginning, monthly later

- Chuprov taught at Petersburg Polytechnic Institute, wrote book "Essays on the Theory of Statistics", 1909

- discuss proper citation and attribution of earlier contributors, banter about scientific merit and correctness of prob./stat. texts of day, edit each other's papers and editions of books.

- Markov writes, "[this] correspondence had some influence on the first chapter of my book."                                        (Dec. 8, 1912 letter to Chuprov)

- plan and participate in Bicentennial Celebration of LLN, honoring J. Bernoulli, "Ars Conjectandi", 1713, held on Dec. 3, 1913.

# No. 62

*(Letter from Markov to Chuprov)*

3 December 1913

Highly respected Alexander Alexandrovich:

In order to avoid any misunderstanding I hasten to inform you that I cannot subscribe to A. V. Vasiliev's project to put together a general collection of the three talks. We joined together only temporarily on the basis of mutual independence. From the outside the celebration appears to have turned out fairly successfully. But there is not complete solidarity between us. In your talk statistics stood first and foremost, and applications of the law of large numbers were advanced that seem questionable to me.

By subscribing to them I can only weaken that which for me is particularly dear: the rigor of judgments I permit.

For example, it is very important for me to observe that Poisson did not prove his theorem; moreover, I cannot consider a statement a theorem unless it is established precisely.

Your talk harmonized beautifully with A. V. Vasiliev's talk but in no way with mine. I had to give my talk since the 200th year of a mathematical theorem was being celebrated, but I do not intend to publish it and I do not wish to. The combination of your talk with A. V. Vasiliev's talk without mine is quite possible, as is clear, by the way, from P. Yushkevich's article in the newspaper *The Day* (of 2 December, No. 327); and I suggest it be done.

Sincerely yours,

A. MARKOV

# ≋ USGS
## Astrogeology Research Program

*GAZETTEER OF PLANETARY NOMENCLATURE*

MOON NOMENCLATURE: CRATER

See **CRATER in the descriptor terms page** for additional information.

| NAME | LAT | LONG | DIAM | CT | ET | MAP | QUAD | AS | AD | REF | FT | ORIGIN |
|------|-----|------|------|----|----|-----|------|-----|-----|-----|-----|--------|
| Abbe | 57.3S | 175.2E | 66.0 | EU | GE | 3 | LMP | 5 | 1970 | 0 | AA | Ernst K.; German optician, physician, astronomer (1840-1905). |
| Abbot | 5.6N | 54.8E | 10.0 | NA | AM | 062D1 | LTO | 5 | 1973 | 0 | AA | Charles Greeley; American astrophysicist (1872-1973). |
| [Abduh] | 14.7N | 39.0E | 9.0 | AF | EG | 061A2 | LTO | 6 | 0 | 0 | AA | Mohammed; Egyptian writer (1849-1905). |
| Marinus | 39.4S | 76.5E | 58.0 | EU | GR | 3 | LOC | 5 | 1935 | 66 | AA | Of Tyre; Greek geographer (unkn-c. 100). |
| Mariotte | 28.5S | 139.1W | 65.0 | EU | FR | 4 | LOC | 5 | 1970 | 0 | AA | Edme; French physicist (1620-1684). |
| Marius | 11.9N | 50.8W | 41.0 | EU | GE | 56 | LAC | 5 | 1935 | 66 | AA | Mayer, Simon; German astronomer (1570-1624). |
| Markov | 53.4N | 62.7W | 40.0 | UR | RU | 3 | LMP | 5 | 1964 | 67 | AA | Aleksandr V.; Soviet astrophysicist (1897-1968); Andrei A.; Russian mathematician (1856-1922). |

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