Updating The PageRank Vector

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The PageRank Vector

Definition

$\boldsymbol{\pi}^T$ = stationary distribution of a Markov chain

$$\mathbf{P} = t\mathbf{T} + (1-t)\mathbf{E} \qquad 0 < t < 1$$

Irreducible & Aperiodic

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Big Eigenvector Problem

Solve $\pi^T = \pi^T \mathbf{P} \qquad \pi^T \mathbf{e} = 1$

$n = O(10^9)$ (too big for direct solves)

"World's Largest Matrix Computation" (Cleve Moler)

Computing π^T

Iterate

Start with $\boldsymbol{\pi}_0^T = \mathbf{e}/n$ and iterate $\boldsymbol{\pi}_{j+1}^T = \boldsymbol{\pi}_j^T \mathbf{P}$ (power method)

Convergence Time

Use to be measured in days

Computing π^T

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Start with $\pi_0^T = \mathbf{e}/n$ and iterate $\pi_{j+1}^T = \pi_j^T \mathbf{P}$ (power method)

Convergence Time

Use to be measured in days

Now ???

Recent Advances

Extrapolation methods for accelerating PageRank, *Kamvar, Haveliwala, Manning, Golub, 03*

Exploiting the block structure of the web for computing PageRank, *K, H, M, Golub, 03*

Adaptive methods for the computation of PageRank, *Kamvar, Haveliwala, Golub, 03*

Partial state space aggregation based on lumpability and its application to PageRank, *Chris Lee, 03*

Updating

Easy Problem

No pages added — No pages removed

- Size does not change — only probabilities change

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No pages added — No pages removed

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Hard Problem

Both pages & links are added or removed

- Both size & probabilities change

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Hard Problem

Both pages & links are added or removed

— Both size & probabilities change

The Trouble

Prior results are not much help

— Google just restarts from scratch every few weeks

Perron Complementation

Perron Frobenius

$$\mathbf{P} \geq 0 \quad \text{irreducible} \quad \Longrightarrow \quad \rho = \rho(\mathbf{P}) \quad \text{simple eigenvalue}$$

Unique Left-Hand Perron Vector

$$\boldsymbol{\pi}^T\mathbf{P} = \rho\,\boldsymbol{\pi}^T \qquad \boldsymbol{\pi}^T > 0 \qquad \|\boldsymbol{\pi}^T\|_1 = 1$$

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Partition

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix}$$

Shift $\mathbf{P}$ by ρ $\longrightarrow$ Schur Complements $\longrightarrow$ Shift back by ρ

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Shift $\mathbf{P}$ by ρ $\longrightarrow$ Schur Complements $\longrightarrow$ Shift back by ρ

Perron Complements

$$\mathbf{S}_1 = \mathbf{P}_{11} + \mathbf{P}_{12}(\rho\,\mathbf{I} - \mathbf{P}_{22})^{-1}\mathbf{P}_{21}$$

$$\mathbf{S}_2 = \mathbf{P}_{22} + \mathbf{P}_{21}(\rho\,\mathbf{I} - \mathbf{P}_{11})^{-1}\mathbf{P}_{12}$$

Inherited Properties

For $\mathbf{P} \geq 0$ irreducible with $\rho = \rho(\mathbf{P})$

$$\mathbf{S}_i \geq 0$$

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$\mathbf{S}_i$ is stochastic

$\mathbf{S}_i$ represents a censored Markov chain

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For $\mathbf{P}$ stochastic

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$\mathbf{S}_i$ represents a censored Markov chain

Censored Perron vectors

$\mathbf{s}_i^T$ = Left-hand Perron vector for $\mathbf{S}_i$

$\mathbf{s}_i^T \mathbf{S}_i = \rho\, \mathbf{s}_i^T$

Aggregation

Objective

Use $\mathbf{s}_1^T \ \mathbf{s}_2^T \ \cdots$ to build $\boldsymbol{\pi}^T$

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Use $\mathbf{s}_1^T \; \mathbf{s}_2^T \; \cdots$ to build $\boldsymbol{\pi}^T$

Aggregation Matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{s}_1^T \mathbf{P}_{11} \mathbf{e} & \mathbf{s}_1^T \mathbf{P}_{12} \mathbf{e} \\ \mathbf{s}_2^T \mathbf{P}_{21} \mathbf{e} & \mathbf{s}_2^T \mathbf{P}_{22} \mathbf{e} \end{bmatrix}_{2 \times 2}$$

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$\rho(\mathbf{A}) = \rho = \rho(\mathbf{P}) = \rho(\mathbf{S}_i)$

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Inherited Properties

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$\mathbf{A}$ is irreducible

$\rho(\mathbf{A}) = \rho = \rho(\mathbf{P}) = \rho(\mathbf{S}_i)$

$\mathbf{P}$ stochastic $\Longrightarrow$ $\mathbf{A}$ stochastic

Disaggregation

The A / D Theorem

If

$$\mathbf{s}_i^T = \text{Perron vectors for } \mathbf{S}_i = \mathbf{P}_{ii} + \mathbf{P}_{i*}(\rho\mathbf{I} - \mathbf{P}_{**})^{-1}\mathbf{P}_{*i}$$

$$\boldsymbol{\alpha}^T = (\alpha_1,\ \alpha_2) = \text{Perron vector for } \mathbf{A} = \begin{bmatrix} \mathbf{s}_1^T\mathbf{P}_{11}\mathbf{e} & \mathbf{s}_1^T\mathbf{P}_{12}\mathbf{e} \\ \mathbf{s}_2^T\mathbf{P}_{21}\mathbf{e} & \mathbf{s}_2^T\mathbf{P}_{22}\mathbf{e} \end{bmatrix}_{2\times 2}$$

then

$$\boldsymbol{\pi}^T = (\,\alpha_1\mathbf{s}_1^T \mid \alpha_2\mathbf{s}_2^T\,) = \text{Perron vector for } \mathbf{P}_{n\times n}$$

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Corollary

$$\mathbf{s}_1^T = (\pi_1, \ldots, \pi_g)/\sum_{i=1}^{g}\pi_i \qquad \mathbf{s}_2^T = (\pi_{g+1}, \ldots, \pi_n)/\sum_{i=g+1}^{n}\pi_i$$

Updating By Aggregation

Prior Data

$\mathbf{Q}_{m\times m}$ = Old Google Matrix (known)

$\boldsymbol{\phi}^T = (\phi_1, \phi_2, \ldots, \phi_m)$ = Old PageRank Vector (known)

Updating By Aggregation

Prior Data

$\mathbf{Q}_{m\times m}$ = Old Google Matrix (known)

$\boldsymbol{\phi}^T = (\phi_1,\ \phi_2,\ \ldots,\ \phi_m)$ = Old PageRank Vector (known)

Updated Data

$\mathbf{P}_{n\times n}$ = New Google Matrix (known)

$\boldsymbol{\pi}^T = (\pi_1,\ \pi_2,\ \ldots,\ \pi_n)$ = New PageRank Vector (unknown)

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Separate Pages Likely To Be Most Affected

$G = \{\text{most affected}\}$ $\quad \overline{G} = \{\text{less affected}\}$ $\quad \mathcal{S} = G \cup \overline{G}$

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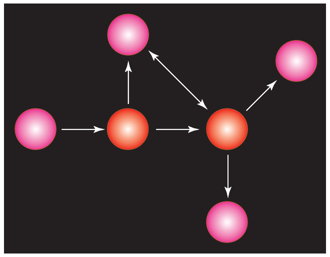
Separate Pages Likely To Be Most Affected

$G = \{\text{most affected}\}$ $\quad \overline{G} = \{\text{less affected}\}$ $\quad \mathcal{S} = G \cup \overline{G}$

New pages (and neighbors) go into G



Aggregation

Partition

$$\mathbf{P}_{n\times n} = \begin{array}{c} \\ G \\ \overline{G} \end{array}\!\!\begin{array}{c} \begin{array}{cc} G & \overline{G} \end{array} \\ \begin{pmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{pmatrix} \end{array} = \left[\begin{array}{ccc|c} p_{11} & \cdots & p_{1g} & \mathbf{r}_1^T \\ \vdots & \ddots & \vdots & \vdots \\ p_{g1} & \cdots & p_{gg} & \mathbf{r}_g^T \\ \hline \mathbf{c}_1 & \cdots & \mathbf{c}_g & \mathbf{P}_{22} \end{array}\right]$$

$$\boldsymbol{\pi}^T = (\pi_1, \ldots \pi_g \mid \pi_{g+1}, \ldots, \pi_n)$$

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Perron Complements

$p_{11} \cdots p_{gg}$ are 1×1 $\Longrightarrow$ Perron complements $= 1$

$\Longrightarrow$ Perron vectors $= 1$

Aggregation

Partition

$$\mathbf{P}_{n\times n} = \begin{matrix} & G & \overline{G} \\ G & \multicolumn{2}{c}{\left(\begin{matrix}\mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22}\end{matrix}\right)} \\ \overline{G} & & \end{matrix} = \left[\begin{array}{c|c|c|c} p_{11} & \cdots & p_{1g} & \mathbf{r}_1^T \\ \hline \vdots & \ddots & \vdots & \vdots \\ \hline p_{g1} & \cdots & p_{gg} & \mathbf{r}_g^T \\ \hline \mathbf{c}_1 & \cdots & \mathbf{c}_g & \mathbf{P}_{22} \end{array}\right]$$

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One significant complement $\mathbf{S}_2 = \mathbf{P}_{22} + \mathbf{P}_{21}(\mathbf{I} - \mathbf{P}_{11})^{-1}\mathbf{P}_{12}$

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One significant Perron vector $\mathbf{s}_2^T\mathbf{S}_2 = \mathbf{s}_2^T$

A/D corollary $\Longrightarrow$ $\mathbf{s}_2^T = (\pi_{g+1}, \ldots, \pi_n) / \sum_{i=g+1}^{n} \pi_i$

Approximate Aggregation

Some Old PageRanks Approximate New Ones

$$(\pi_{g+1}, \ldots, \pi_n) \approx (\phi_{g+1}, \ldots, \phi_n)$$

(the smaller ones)

By A/D Corollary

$$\mathbf{s}_2^T = \frac{(\pi_{g+1}, \ldots, \pi_n)}{\sum_{i=g+1}^{n} \pi_i} \approx \frac{(\phi_{g+1}, \ldots, \phi_n)}{\sum_{i=g+1}^{n} \phi_i} \equiv \widetilde{\mathbf{s}}_2^T$$

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Approximate Aggregation Matrix

$$\widetilde{\mathbf{A}} \equiv \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12}\mathbf{e} \\ \widetilde{\mathbf{s}}_2^T \mathbf{P}_{21} & \widetilde{\mathbf{s}}_2^T \mathbf{P}_{22}\mathbf{e} \end{bmatrix}_{g+1 \times g+1}$$

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$$\widetilde{\boldsymbol{\alpha}}^T = \left(\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_g, \widetilde{\alpha}_{g+1} \right)$$

By A/D Theorem

$$\widetilde{\boldsymbol{\pi}}^T \equiv \left(\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_g \,\middle|\, \widetilde{\alpha}_{g+1} \widetilde{\mathbf{s}}_2^T \right) \approx \boldsymbol{\pi}^T$$

(not bad)

Iterative Aggregation

Improve By Successive Aggregation / Disaggregation?

NO! Can’t do twice — fixed point emerges

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Solution

Perturb A/D output to move off of fixed point

Move in direction of solution

$\widetilde{\widetilde{\boldsymbol{\pi}}}^T = \widetilde{\boldsymbol{\pi}}^T \mathbf{P}$ (a smoothing step)

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NO! Can't do twice — fixed point emerges

Solution

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Move in direction of solution

$\tilde{\tilde{\pi}}^T = \tilde{\pi}^T \mathbf{P}$ (a smoothing step)

The Iterative A/D Updating Algorithm

Determine the "G-set" partition $\mathcal{S} = G \cup \overline{G}$

Approximate A/D step generates $\tilde{\pi}^T$

Smooth $\tilde{\tilde{\pi}}^T = \tilde{\pi}^T \mathbf{P}$

Use $\tilde{\tilde{\pi}}^T$ as input to another approximate aggregation step

$\vdots$

Convergence

THEOREM

Always converges to the new PageRank vector $\boldsymbol{\pi}^T$

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Rate of convergence governed by $|\lambda_2(\mathbf{S}_2)|$

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THE GAME

Find a relatively small G to minimize $|\lambda_2(\mathbf{S}_2)|$

Experiments

Test Networks From Crawl Of Web

(Supplied by Ronny Lempel)

Censorship
562 nodes 736 links

Movies
451 nodes 713 links

MathWorks (Supplied by Cleve Moler)
517 nodes 13,531 links

Abortion
1,693 nodes 4,325 links

Genetics
2,952 nodes 6,485 links

California
9,664 nodes 16,150 links

Perturbations

The Updates

Nodes Added = 3

Nodes Removed = 50

Links Added = 10 (Different values have little effect on results)

Links Removed = 20

Stopping Criterion

1-norm of residual $< 10^{-10}$

Movies

Power Method

Iterations	Time
17	.40

Iterative Aggregation

$\lvert G \rvert$	Iterations	Time
5	12	.39
10	12	.37
15	11	.36
20	11	.35
100	9	.33
200	8	.35
300	7	.39
400	6	.47

$nodes = 451 \quad links = 713$

Movies

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$\|G\|$	Iterations	Time
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$nodes = 451 \quad links = 713$

Censorship

Power Method

Iterations	Time
38	1.40

Iterative Aggregation

$\lvert G \rvert$	Iterations	Time
5	38	1.68
10	38	1.66
15	38	1.56
20	20	1.06
25	20	1.05
50	10	.69
100	8	.55
300	6	.65
400	5	.70

$nodes = \mathbf{562} \quad links = \mathbf{736}$

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400	5	.70

$nodes = \mathbf{562} \quad links = \mathbf{736}$

MathWorks

Power Method

Iterations	Time
$\mathbf{54}$	$\mathbf{1.25}$

Iterative Aggregation

$\|G\|$	Iterations	Time
$\mathbf{5}$	$\mathbf{53}$	$\mathbf{1.18}$
$\mathbf{10}$	$\mathbf{52}$	$\mathbf{1.29}$
$\mathbf{15}$	$\mathbf{52}$	$\mathbf{1.23}$
$\mathbf{20}$	$\mathbf{42}$	$\mathbf{1.05}$
$\mathbf{25}$	$\mathbf{20}$	$\mathbf{1.13}$
$\mathbf{300}$	$\mathbf{11}$	$\mathbf{.83}$
$\mathbf{400}$	$\mathbf{10}$	$\mathbf{1.01}$

$nodes = \mathbf{517} \quad links = \mathbf{13,531}$

MathWorks

Power Method

Iterations	Time
54	1.25

Iterative Aggregation

$\lvert G \rvert$	Iterations	Time
5	53	1.18
10	52	1.29
15	52	1.23
20	42	1.05
25	20	1.13
50	18	.70
100	16	.70
200	13	.70
300	11	.83
400	10	1.01

$nodes = \mathbf{517} \quad links = \mathbf{13,531}$

Abortion

Power Method

Iterations	Time
106	37.08

Iterative Aggregation

$\|G\|$	Iterations	Time
5	109	38.56
10	105	36.02
15	107	38.05
20	107	38.45
25	97	34.81
50	53	18.80
250	12	5.62
500	6	5.21
750	5	10.22
1000	5	14.61

$nodes = \mathbf{1,693} \quad links = \mathbf{4,325}$

Abortion

Power Method

Iterations	Time
106	37.08

Iterative Aggregation

$\|G\|$	Iterations	Time
5	109	38.56
10	105	36.02
15	107	38.05
20	107	38.45
25	97	34.81
50	53	18.80
100	13	5.18
250	12	5.62
500	6	5.21
750	5	10.22
1000	5	14.61

$nodes = 1,693 \quad links = 4,325$

Genetics

Power Method

Iterations	Time
92	91.78

Iterative Aggregation

$\lvert G \rvert$	Iterations	Time
5	91	88.22
10	92	92.12
20	71	72.53
50	25	25.42
100	19	20.72
250	13	14.97
1000	5	17.76
1500	5	31.84

$nodes = \mathbf{2,952} \quad links = \mathbf{6,485}$

Genetics

Power Method

Iterations	Time
92	91.78

Iterative Aggregation

$\lvert G \rvert$	Iterations	Time
5	91	88.22
10	92	92.12
20	71	72.53
50	25	25.42
100	19	20.72
250	13	14.97
500	7	11.14
1000	5	17.76
1500	5	31.84

$nodes = \mathbf{2,952} \quad links = \mathbf{6,485}$

California

Power Method

Iterations	Time
176	5.85

Iterative Aggregation

$\|G\|$	Iterations	Time
500	19	1.12
1000	15	.92
1250	20	1.04
2000	13	1.17
5000	6	1.25

$nodes = \mathbf{9,664} \quad links = \mathbf{16,150}$

California

Power Method

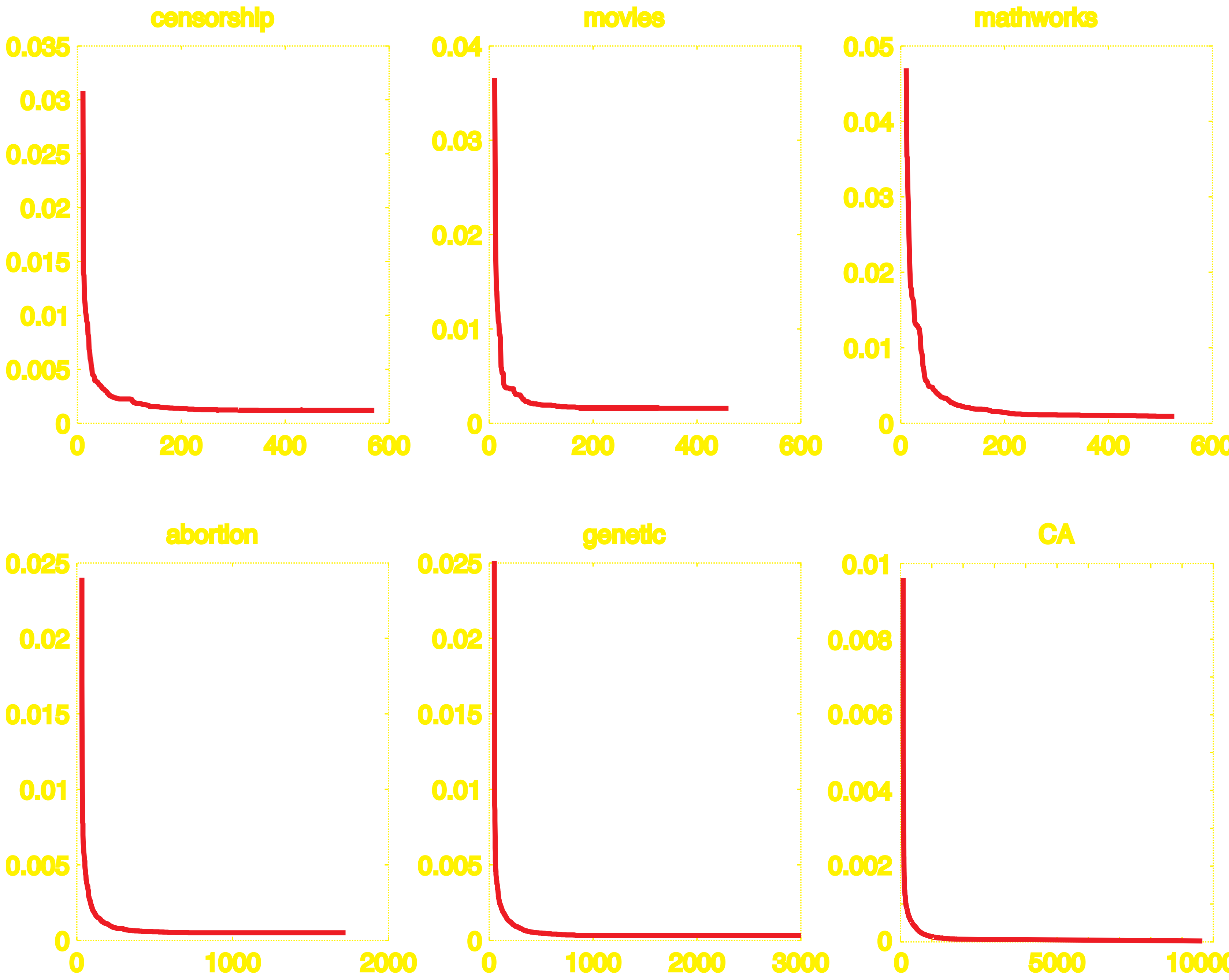
Iterations	Time
176	5.85

Iterative Aggregation

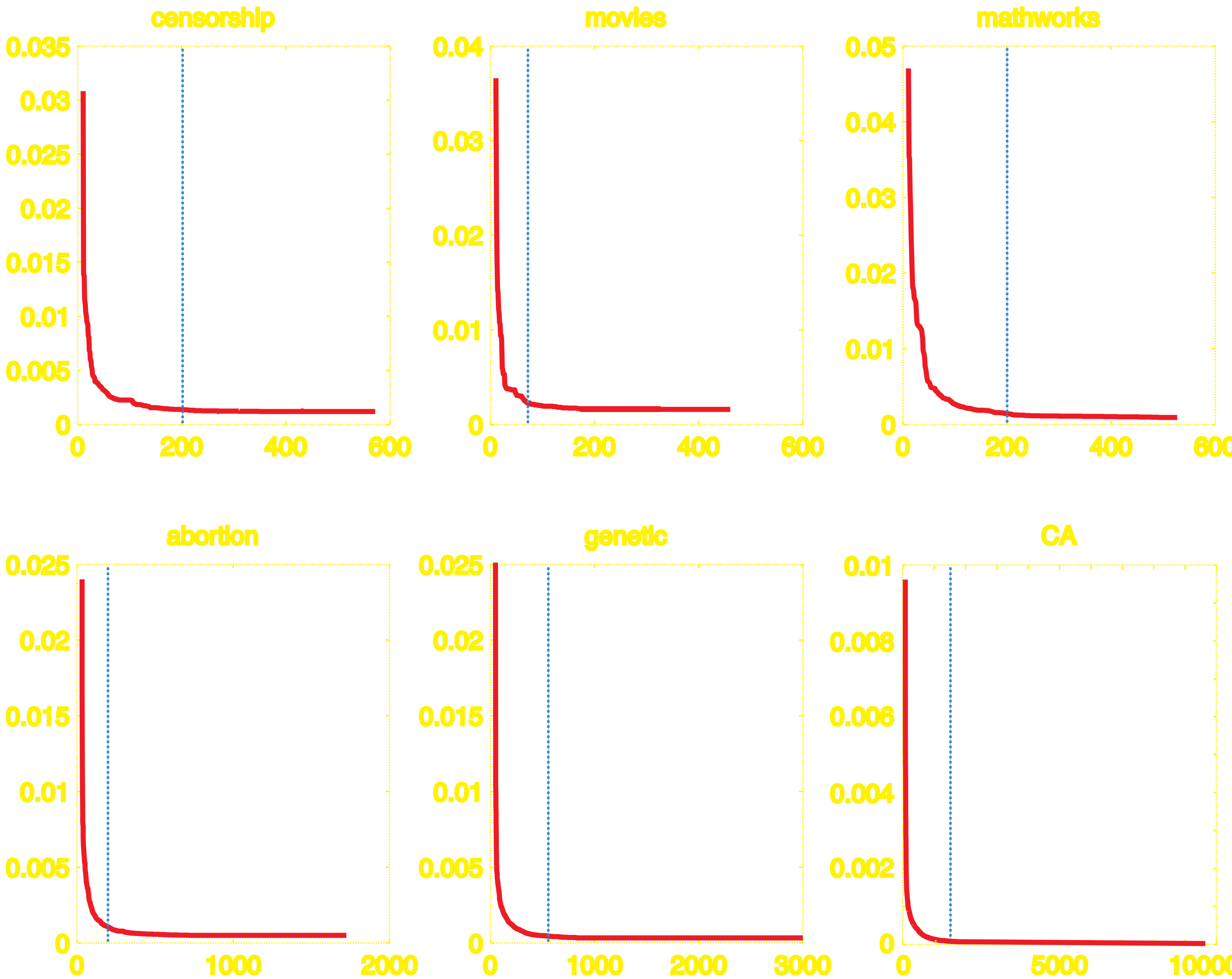
$\lvert G \rvert$	Iterations	Time
500	19	1.12
1000	15	.92
1250	20	1.04
1500	14	.90
2000	13	1.17
5000	6	1.25

$nodes = \mathbf{9,664} \quad links = \mathbf{16,150}$

"L" Curves


censorship
movies
mathworks
abortion
genetic
CA
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
0.04
0.05
0.008
0.006
0.004
0.002
200
400
600
1000
2000
3000
5000
10000

"L" Curves


censorship
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
0
200
400
600
movies
0.04
0.03
0.02
0.01
0
0
200
400
600
mathworks
0.05
0.04
0.03
0.02
0.01
0
0
200
400
600
abortion
0.025
0.02
0.015
0.01
0.005
0
0
1000
2000
genetic
0.025
0.02
0.015
0.01
0.005
0
0
1000
2000
3000
CA
0.01
0.008
0.006
0.004
0.002
0
0
5000
10000

Comparisons

Race

- → **Power Method**
- → **Power Method + Quadratic Extrapolation**
- → **Iterative Aggregation**
- → **Iterative Aggregation + Quadratic Extrapolation**

Comparisons

Race

→ **Power Method**

→ **Power Method + Quadratic Extrapolation**

→ **Iterative Aggregation**

→ **Iterative Aggregation + Quadratic Extrapolation**

NC State Internal Crawl

→ **10,000 nodes + 101,118 links**

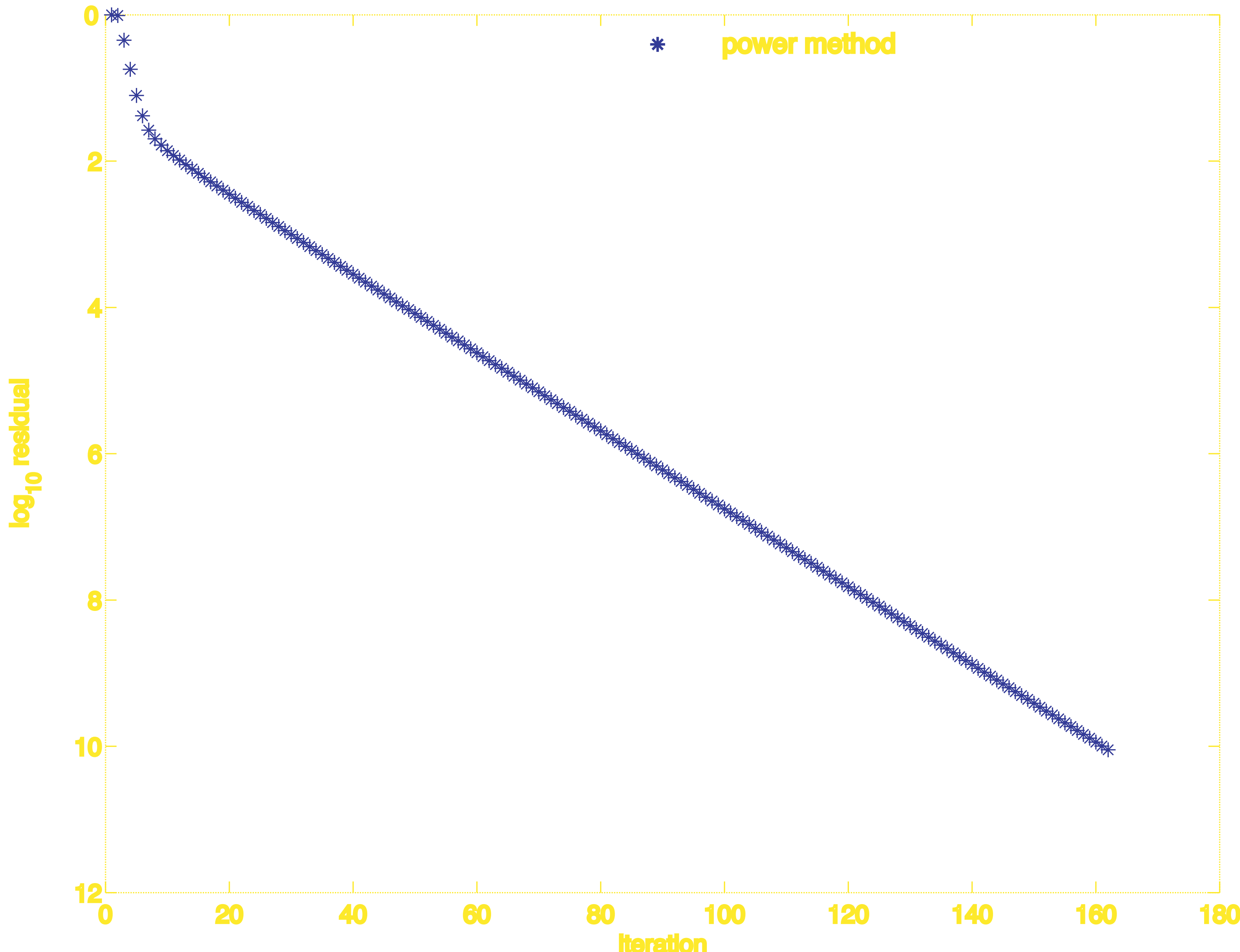
→ 50 **nodes added**

→ 30 **nodes removed**

→ 300 **links added**

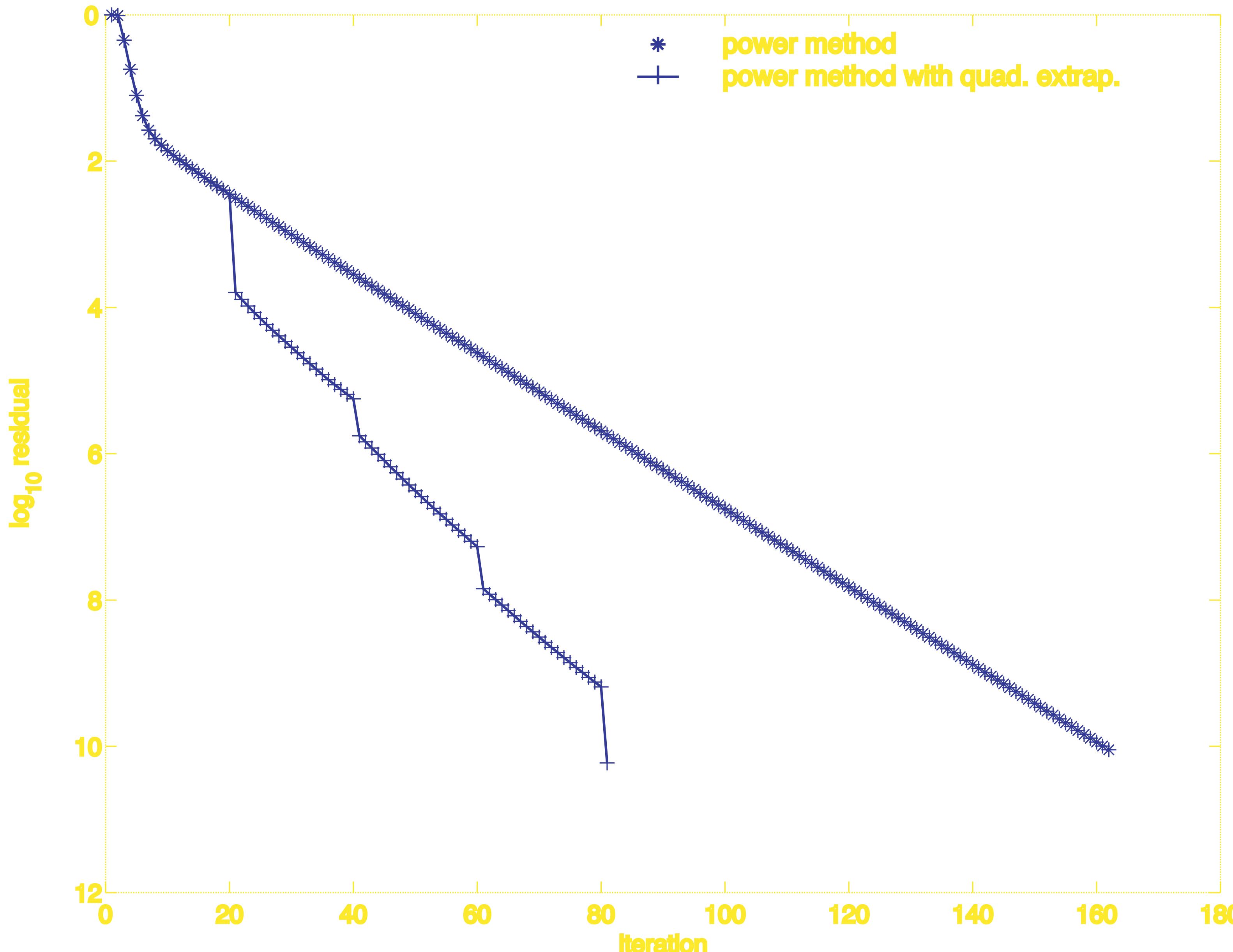
→ 200 **links removed**

Iterations

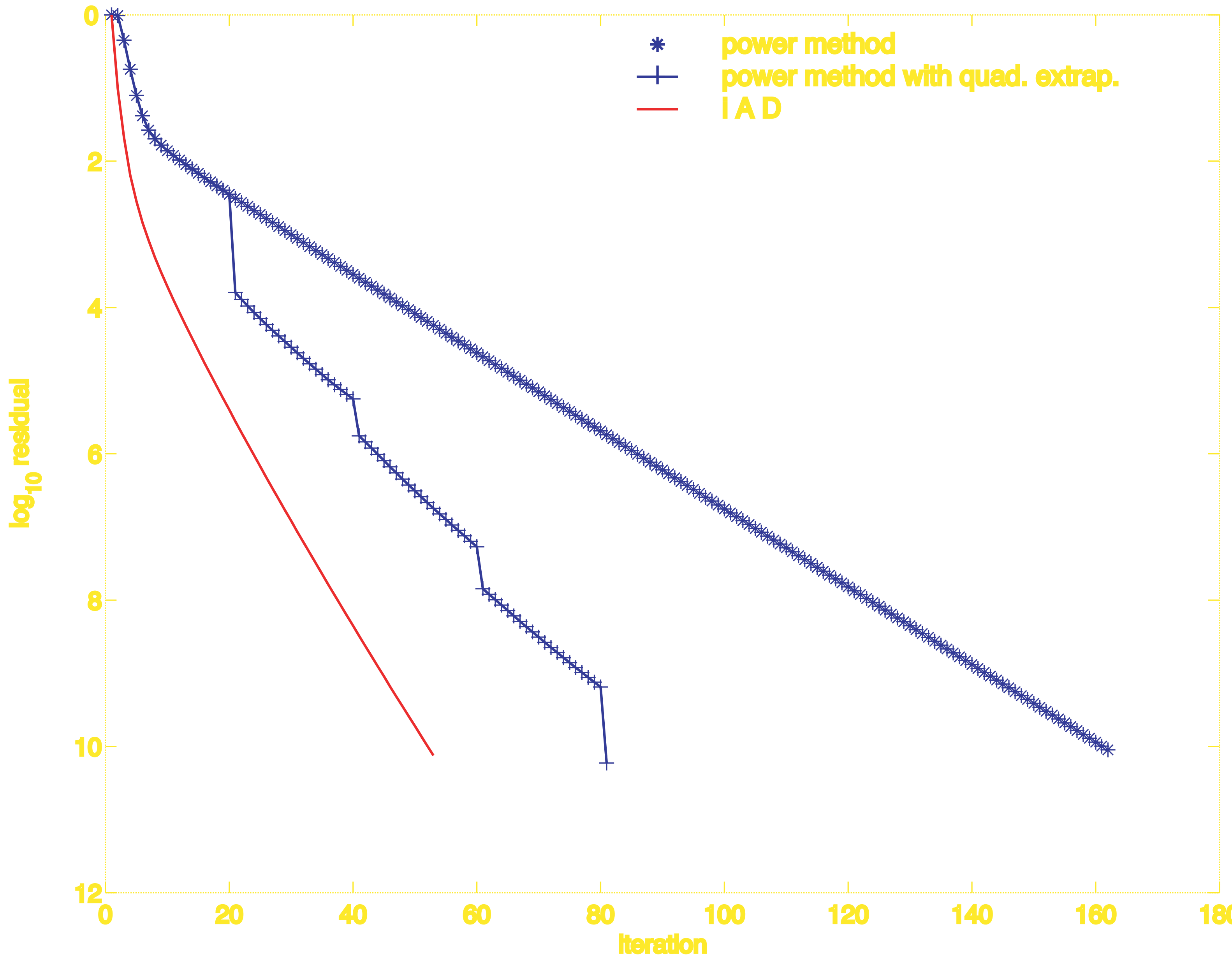


power method
log10 residual
0
2
4
6
8
10
12
0
20
40
60
80
100
120
140
160
180
Iteration

Iterations

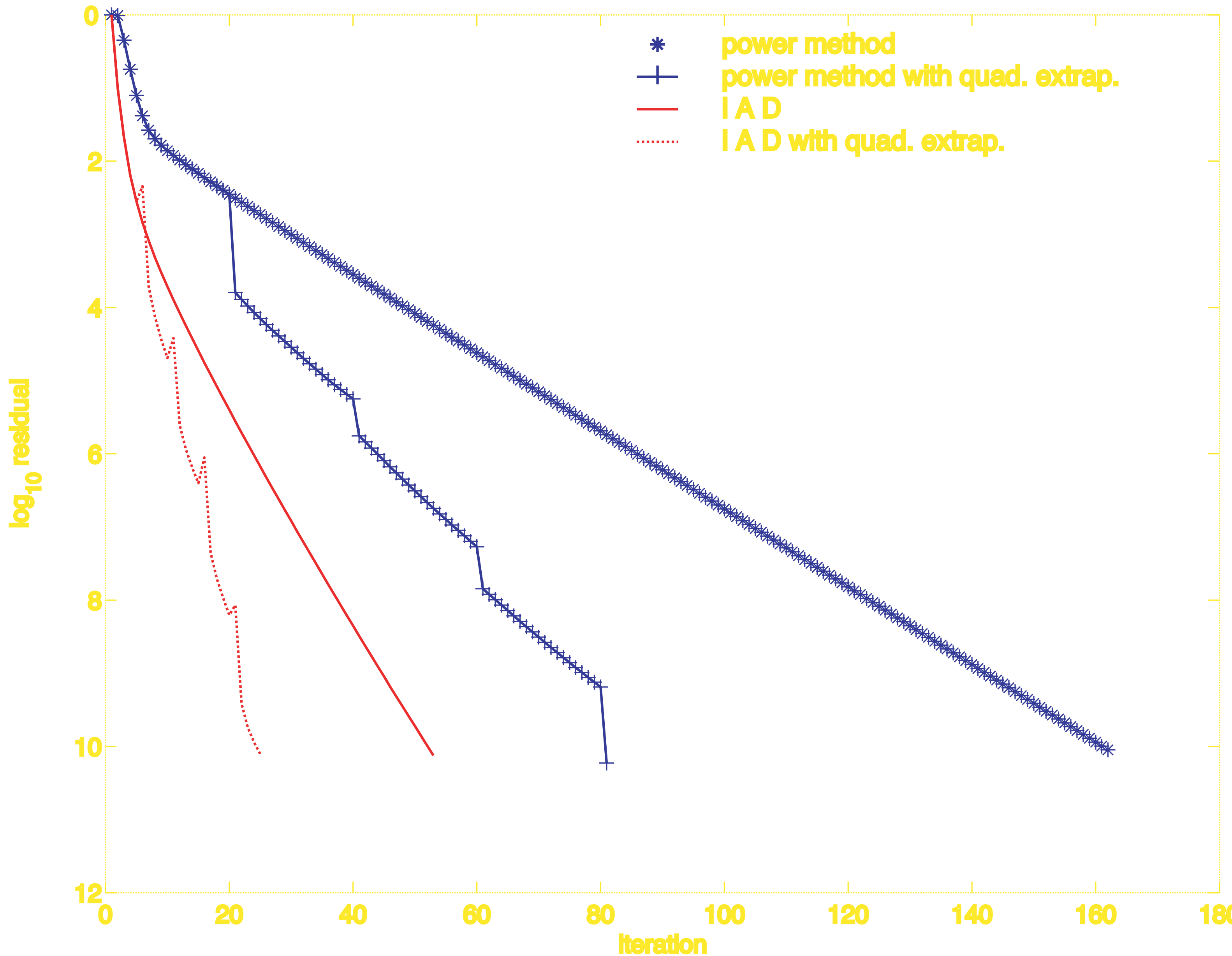

power method
power method with quad. extrap.
log10 residual
Iteration
0
2
4
6
8
10
12
0
20
40
60
80
100
120
140
160
180

Iterations



power method
power method with quad. extrap.
I A D
0
2
4
6
8
10
12
0
20
40
60
80
100
120
140
160
180
Iteration
log10 residual

Iterations



power method
power method with quad. extrap.
I A D
I A D with quad. extrap.
0
2
4
6
8
10
12
0
20
40
60
80
100
120
140
160
180
Iteration
$\log_{10}$ residual

Timings

	Iterations	Time (sec)	$\|G\|$
Power	$\mathbf{162}$	$\mathbf{9.69}$	
Power+Quad			
IAD			
IAD+Quad			

$nodes = \mathbf{10,000} \quad links = \mathbf{101,118}$

Timings

	Iterations	Time (sec)	$\lvert G \rvert$
Power	$\mathbf{162}$	$\mathbf{9.69}$	
Power+Quad	$\mathbf{81}$	$\mathbf{5.93}$	
IAD			
IAD+Quad			

$nodes = \mathbf{10,000} \quad links = \mathbf{101,118}$

Timings

	Iterations	Time (sec)	$\|G\|$
Power	162	9.69	
Power+Quad	81	5.93	
IAD	21	2.22	2000
IAD+Quad			

$nodes = 10,000 \quad links = 101,118$

Timings

	Iterations	Time (sec)	$\|G\|$
Power	162	9.69	
Power+Quad	81	5.93	
IAD	21	2.22	2000
IAD+Quad	16	1.85	2000

$nodes = 10,000 \quad links = 101,118$

Conclusion

- Iterative A / D with appropriate partitioning and smoothing shows promise for updating Markov chains with power law distributions