# Mathematical Fuel For Search Engines 

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"The internet without search is like a cruise" missile without a guidance system:

Bob Davis, CEO, LYCOS
= Asia
Green = Europe Blue $=$ North America Yellow = Latin America

White = Other Cyan = Private Internets

## Search Engines

## 

## Harvard 1962 - 1965

IBM 7094 \& IBM 360

## Gerard Salton

Implemented at Cornell (1965-1970)
Based on matrix methods

## Term-Document Matrices

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## Term-Document Matrix

$$
\left.\begin{array}{ccccc} 
& \text { Doc } 1 & \text { Doc } 2 & \cdots & \text { Doc } n \\
\text { Term 1 } \\
\text { Term 2 } \\
\vdots \\
\text { Term m } & f_{11} & f_{12} & \cdots & f_{1 n} \\
f_{21} & f_{22} & \cdots & f_{2 n} \\
\vdots & \vdots & \ddots & \vdots \\
f_{m 1} & f_{m 2} & \cdots & f_{m n}
\end{array}\right)=\boldsymbol{A}_{m \times n}
$$

## Query Matching

## Query Vector

$$
\mathbf{q}^{T}=\left(q_{1}, q_{2}, \ldots, q_{m}\right) \quad q_{i}= \begin{cases}1 & \text { if Term } i \text { is requested } \\ 0 & \text { if not }\end{cases}
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\text { Use } \delta_{i}=\cos \theta_{i}=\frac{\mathbf{q}^{T} \mathbf{A}_{i}}{\|\mathbf{q}\|\left\|\mathbf{A}_{i}\right\|}
\end{array}
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Rank documents by size of $\delta_{i}$

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Rank documents by size of $\delta_{i}$
Return Document $i$ to user when $\delta_{i} \geq$ tol

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## A Problem

Suppose query = NCSU
Suppose $N C S U$ occurs once in $D_{1}$ and twice in $D_{2}$

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$$
\text { Set } q_{i}= \begin{cases}\log \left(n / \nu_{i}\right) & \text { if } \nu_{i} \neq 0 \\ 0 & \text { if } \nu_{i}=0\end{cases}
$$

## Uncertainties

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Variation in Indexing Conventions

- No two people index documents the same way
- Computer indexing is inexact and can be unpredictable


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In Theory — it's simple and elegant

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- $D_{1}$ indexed by gas, car, tire


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- $D_{2}$ indexed by automobile, fuel, and tire

The Challenge

- Find $D_{2}$ by revealing the latent connection through tire


## Latent Semantic Indexing

Use a Fourier expansion of A

$$
\mathbf{A}=\sum_{i=1}^{r} \sigma_{i} \mathbf{Z}_{i}, \quad\left\langle\mathbf{Z}_{i} \mid \mathbf{Z}_{j}\right\rangle=\left\{\begin{array}{ll}
1 & i=j, \\
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Realign data along dominant directions $\left\{\mathbf{Z}_{1}, \ldots, \mathbf{Z}_{k}, \mathbf{Z}_{k+1}, \ldots, \mathbf{Z}_{r}\right\}$

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- Haar: $\mathbf{A}=\mathbf{H}_{m} \mathbf{B} \mathbf{H}_{n}^{T}=\sum_{i, j} \beta_{i j} \mathbf{h}_{i} \mathbf{h}_{j}^{T}$
( h's only use -1, 0, 1 )


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- Doesn't scale up well

Impractical for www

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Direct Query Matching

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Return $P_{i}, P_{j}, P_{k}, \ldots$ in order of ranks $r_{i}, r_{j}, r_{k}, \ldots$

## How To Measure "Importance"



Hubs


## How To Measure "Importance"



Hubs


Good hub pages point to good authority pages

## How To Measure "Importance"



Good hub pages point to good authority pages

- Good authorities are pointed to by good hubs


## HITS Algorithm

Hypertext Induced Topic Search
(J. Kleinberg 1998)

## Determine Authority \& Hub Scores

- $a_{i}=$ authority score for $P_{i}$ - $h_{i}=$ hub score for $P_{i}$


## HITS Algorithm

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Successive Refinement

- Start with $h_{i}(0)=1$ for all pages $P_{i}$


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Successive Refinement

- Start with $h_{i}(0)=1$ for all pages $P_{i}$
- Successively refine rankings
- For $k=1,2, \ldots$

$$
a_{i}(k)=\sum_{j: P_{j} \rightarrow P_{i}} h_{j}(k-1)
$$

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$$
\mathbf{L}_{i j}= \begin{cases}1 & P_{i} \rightarrow P_{j} \\ 0 & P_{i} \nrightarrow P_{j}\end{cases}
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a_{i}(k)=\sum_{j: P_{j} \rightarrow P_{i}} h_{j}(k-1) \Rightarrow \mathbf{a}_{k}=\mathbf{L}^{T} \mathbf{h}_{k-1}
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2. Build neighborhood graph


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## Advantages

- Returns satisfactory results


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- If Yahoo! "votes" for $n$ pages
- then $P$ receives only $r(Y) / n$ credit from $Y$


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r(P)=\sum_{P \in \mathcal{B}_{P}} \frac{r(P)}{|P|}
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$\mathcal{B}_{P}=\{$ all pages pointing to $P\}$
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& \bullet \\
& r_{j+1}\left(P_{i}\right)=\sum_{P \in \mathcal{B}_{P_{i}}} \frac{r_{j}(P)}{|P|}
\end{aligned}
$$

## In Matrix Notation

After Step $j$

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\pi_{j}^{T}=\left[r_{j}\left(P_{1}\right), r_{j}\left(P_{2}\right), \cdots, r_{j}\left(P_{n}\right)\right]
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(provided limit exists)

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$\boldsymbol{\pi}_{j+1}^{T}=\boldsymbol{\pi}_{j}^{T} \mathbf{P} \quad$ is random walk on the graph defined by links
$\pi^{T}=\lim _{j \rightarrow \infty} \pi_{j}^{T}=$ steady-state probability distribution

## Random Surfer

## Web Surfer Randomly Clicks On Links

Long-run proportion of time on page $P_{i}$ is $\pi_{i}$

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Replace $\mathbf{P}$ by $\widetilde{\mathbf{P}}=\alpha \mathbf{P}+(1-\alpha) \mathbf{E} \quad e_{i j}=1 / n \quad \alpha \approx .85$

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Different $\mathbf{E}=\mathbf{e v}^{T}$ and $\alpha$ allow customization \& speedup

## THE WALL STREET.JorRalal

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## What's News-

Business and Finance

NEWS CORP, and Liberty are no longer working together on a joint offer to take control of Hughes, with News Corp. proceeding on its own and Liberty considering an independent bid. The move threatens to cloud the process of finding a new owner for the GM unit.
(Arricie on Page A3)

- The SEC signaled it may file civil charges against Morgan Stanley, alleging it doled out IP0 shares based partly on investors' commitments to buy more stock.
(arricle on Page C1)
- Ahold's problems deepened as U.S. authorities opened inquiries into accounting at the Dutch company's U.S. Foodservice unit. - Fleming said the SEC upgraded to a formal investigation an inquiry into the food wholesaler's trade practices with suppliers.
(Articles on Page Az)
- Consumer confidence fell to its lowest level since 1993, hurt by energy costs, the terrorism threat and a stagnant job market.

$$
\begin{aligned}
& \text { (Article on Page A3) } \\
& * * * *
\end{aligned}
$$

## World-Wide

■ BUSH IS PREPARING to present Congress a huge bill for Iraq costs. The total could run to $\$ 95$ billion depending on the length of the possible war and occupation. As horsetrading began at the U.N. to win support for a war resolution, the president again made clear he intends to act with or without the world body's imprimatur. Arms inspectors said Baghdad provided new data, including a report of a possible biological bomb. Gen. Franks assumed command of the war-operations center in Qatar. Allied warplanes are aggressively taking out missile sites that could threaten the allied troop buildup. (Column 4 and Pages A4 and A6) Turkey's parllament debated legislation to let the U.S. deploy 63,000 to open a northem front. Kurdish soldiers lined roads in a show of force as U.S. officials traveled into Iraq's north for an opposition conference.

- Powell said North Korea hasn't restarted a reactor and plutonium-processing facility at Yongbyon, hinting such forbearance might constitute an overture. But saber rattling continued a day after a missile test timed for the inauguration in Seoul. Pyongyang accused U.S. spy planes of violating its airspace and told its army to prepare for U.S. attack. (Page A14)
- The FBI came under withering bipartisan criticism in a Senate Judi-


## Web Master

## As the Web spreads...

Total Internet users, by household, in milions


## Google's U.S. presence expands

Top search engines, in millions Top shopping-referral sites, of unique visitors ${ }^{1}$
in miliions of referrals ${ }^{2}$

| Google |  | Google |  |
| :---: | :---: | :---: | :---: |
|  | 39.4 |  | 12.61 |
| Yahoo Search |  | Deallime |  |
|  | 38.6 | 2.50 |  |
| MSN Search |  | BizRate |  |
|  | 36.8 | 1.93 |  |
| AOL Search |  | Overture |  |
|  | 22.0 | 1.04 |  |
| Ask Jeeves |  | Epinions$0.78$ | ${ }^{2}$ Number of people the sitas send to |
| 13.3 | Hedoding visitus |  |  |
| Overture | fom home and | CNET | stores, induding |
| 6.4 | wort, it lanuary $2003$ | 0.76 | only visitors from nome, for Q 42002 |

## Bush to Seek up to $\$ 95$ Billion To Cover Costs of War on Iraq

By Greg Jaffe And John D. McKinnon

WASHINGTON-The Bush administration is preparing supplemental spending requests totaling as much as $\$ 95$ billion for a war with Iraq, its aftermath and new expenses to fight terrorism, officials said.
The total could be as low as $\$ 60$ billion because Pentagon budget planners don't know how long a military conflict will last, whether U.S. allies will contribute more than token sums to the effort and what damage Saddam Hussein might do
to his own country to retaliate against conquering forces.

Budget planners also are awaiting the outcome of an intense internal debate over whether to include $\$ 13$ billion in the requests to Congress that the Pentagon says it needs to fund the broader war on terrorism, as well as for stepped up homeland security. The White House Office of Management and Budget argues that the money might not be necessary. President Bush, Defense Secretary Donald Rumsfeld and budget director Mitchell Daniels Jr. met yesterday to discuss the matter but didn't reach a final agreement. Mr.

## Cat and Mouse

> As Google Becomes Web's Gatekeeper, Sites Fight to Get In

Search Engine Punishes Firms That Try to Game System; Outlawing the 'Link Farms' Exoticleatherwear Gets Cut Off

## By Michael Totty

And Mylene Mangalindan
Joy Holman sells provocative leather clothing on the Web. She wants what nearly everyone doing business online wants: more exposure on Google.

So from the time she launched exoticleatherwear.com last May, she tried all sorts of tricks to get her site to show up among the first listings when a user of Google Inc.'s popular search engine typed in "women's leatherwear" or "leather apparel." She buried hidden words in her Web pages intended to fool Google's computers. She signed up with a service that promised to have hundreds of sites link to her online store-thereby
boosting a crucial measure in Google's system of ranking sites.

The techniques


## Web Sites Fight for Prime Real Estate on Google

Continued From First Page advertising that tried to capitalize on Google's formula for ranking sites. In ef'fect, SearchKing was offering its clients a chance to boost their own Google rankings by buying ads on more-popular sites. SearchKing filed suit against the search company in federal court in Oklahoma, claiming that Google "purposefully devalued" SearchKing and its customers, damaging its reputation and hurting its advertising sales.

Google won't comment on the case. In court filings, the company said SearchKing "engaged in behavior that would lower the quality of Google search re sults" and alter the company's ranking system.

Google, a closely held company founded by Stanford University graduate students Sergey Brin and Larry Page, says Web companies that want to rank high should concentrate on improving their Web pages rather than gaming its system. "When people try to take scoring into their own hands, that turns into a worse experience for users," says Matt Cutts, a Google software engineer.

## Coding Trickery

Efforts to outfox the search engines have been around since search engines first became popular in the early 1990s. Early tricks included stuffing thousands of widely used search terms in hidden coding, called "metatags." The coding fools a search engine into identifying a site with popular words and phrases that may not actually appear on the site.

Another gimmick was hiding words or terms against a same-color background. The hidden coding deceived search engines that relied heavily on the number of times a word or phrase appeared in ranking a site. But Google's system, based on links, wasn't fooled.

Mr . Brin, 29, one of Google's two founders and now its president of technology, boasted to a San Francisco searchengine conference in 2000 that Google wasn't worried about having its results clogged with irrelevant results because its search methods couldn't be manipulated.

That didn't stop search optimizers from finding other ways to outfox the system. Attempts to manipulate Google's results even became a sport, called Goo-
creating Web sites that were nothing more than collections of links to the clients' site, called "link farms." Since Google ranks a site largely by how many links or "votes" it gets, the link farms could boost a site's popularity.

In a similar technique, called a link exchange, a group of unrelated sites would agree to all link to each other, thereby fooling Google into thinking the sites have a multitude of votes. Many sites also found they could buy links to themselves to boost their rankings.

Ms. Holman, the leatherwear retailer, discovered the consequences of trying to fool Google. The 42 -year-old hospital laboratory technician, who learned computer skills by troubleshooting her hospital's

## 'The big search <br> engines determine the laws of how commerce runs,' says Mr. Massa.

equipment, operates her online apparel store as a side business that she hopes can someday replace her day job.

When she launched her Exotic Leather Wear store from her home in Mesa, Ariz., she quickly learned the importance of appearing near the top of search-engine results, especially on Google. She boned up on search techniques, visiting online discussion groups dedicated to search engines and reading what material she could find on the Web.
At first, Ms. Holman limited herself to modest changes, such as loading her page with hidden metatag coding that would help steer a search toward her site when a user entered words such as "haltertops" or "leather miniskirts." Since Google doesn't give much weight to metatags in determining its rankings, the efforts had little effect on her search results.

She then received an e-mail adver tisement from AutomatedLinks.com, a Wirral, England, company that promised to send traffic "through the roof" by linking more than 2,000 Web sites to hers. Aside from attracting customers. the links were designed to improve her

In theory, when Google encounters the AutomatedLinks code, it treats it as a legitimate referral to the other sites and counts them in toting up the sites' popularity.

Shortly after Ms. Holman signed up with AutomatedLinks in July, she read on an online discussion group that Google objected to such link arrangements. She says she immediately stripped the code from her Web pages. For a while her site gradually worked its way up in Google search results, and business steadily improved because links to her site still remained on the sites of other AutomatedLinks customers. Then, sometime in November, her site was suddenly no longer appearing among the top results. Her orders plunged as much as $80 \%$.

Ms. Holman, who e-mailed Google and AutomatedLinks, says she has been unable to get answers. But in the last few months, other AutomatedLinks customers say they have seen their sites apparently penalized by Google. Graham McLeay, who runs a small chauffeur service north of London, saw revenue cut in half during the two months he believes his site was penalized by Google.

The high-stakes fight between Google and the optimizers can leave some Website owners confused. "I don't know how people are supposed to judge what is right and wrong," says Mr. McLeay.

AutomatedLinks didn't respond to requests for comment. Google declined to comment on the case. But Mr. Cutts, the Google engineer, warns that the rules are clear and that it's better to follow them rather than try to get a problem fixed after a site has been penalized. "We want to return the most relevant pages we can," Mr Cutts says. "The best way for a site owner to do that is follow our guidelines."

## Crackdown

Google has been stepping up its enforcement since 2001. It warned Webmasters that using trickery could get their sites kicked out of the Google Index and it provided a list of forbidden activities, including hiding text and "link schemes," such as the link farms. Google also warned against "cloaking"-showing a search engine a page that's designed to score well while giving visitors a different, more attractive page-or creating multiple Web addresses that take visitors to a single site.
homa City-based SearchKing, an online directory for hundreds of small, specialty Web sites. SearchKing also sells advertising links designed both to deliver traffic to an advertiser and boost its rankings in Google and other search results.

Bob Massa, SearchKing's chief executive, last August launched the PR Ad Network as a way to capitalize on Google's page-ranking system, known as PageRank. PageRank rates Web sites on a scale of one to 10 based on their popularity, and the rankings can be viewed by Web users if they install special Google software. PR Ad Network sells ads that are priced according to a site's PageRank, with higher-ranked sites commanding higher prices. When a site buys an advertising link on a highly ranked site, the ad buyer could see its ratings improve because of the greater weight Google gives to that link.

Shortly after publicizing the ad network, Mr. Massa discovered that his site suddenly dropped in Google's rankings. What's more, sites that participated in the separate SearchKing directory also had their Google rankings lowered. He filed a lawsuit in Oklahoma City federal court, claiming Google was punishing him for trying to profit from the company's page-ranking system.

A Google spokesman won't comment on the case. In its court filings, Google said it demoted pages on the SearchKing site because of SearchKing's attempts to manipulate search results. The company has asked for the suit to be dismissed, arguing that the PageRank represents its opinion of the value of a Web site and as such is protected by the First Amendment.
"The big search engines determine the laws of how commerce runs," says Mr. Massa, who is persisting with the lawsuit even though the sites have had their page rankings partly restored. "Someone needs to demand accountability."

Google is taking steps that many say could satisfy businesses trying to boost their rankings. Google has long sold sponsored links that show up on the top of many search-results pages, separate from the main listings. Last year, the company expanded its paid-listings program, so that there are now more slots where sites can pay for a prominent place in the results. Many sites now are turning to advertising instead of tactics

Hoтe Depo Amid First

By Chad Terhun
ATLANTA-Home Depot I fiscal fourth-quarter earnin $3.4 \%$ on disappointing sales.

Speaking to investors ar analysts, the company's ch chief executive, Bob Nar Home Depot is prepared t dissatisfied customers and competitive challenge from val with remodeled stores, it ventory and improved custor

The nation's largest ho ment retailer said net income ter ended Feb. 2 decreased to or 30 cents a share, from $\$ 71$ 30 cents a share, a year earli $2 \%$ to $\$ 13.21$ billion from $\$ 13.4$ first quarterly sales decline ir ny's 24 -year history. Home the latest quarter was a week a year earlier. Using compar periods, the company said qu increased $5 \%$ and net income

Same-store sales, or sal open at least a year, decline quarter. Home Depot said st last month offset a disastrou and helped the retailer avoi estimate that same-store sale as much as $10 \%$. In 4 p.m Stock Exchange composite tr Depot shares rose 66 cents

## Fiat Patria Is Set to Bec

By Alessandra GA
ROME-Umberto Agnelli named Fiat SpA chairman on ping into the driver's seat as th glomerate works on an 11th-h ing of its unprofitable car un

Mr. Agnelli, the 68-year-0 Fiat patriarch Gianni Agne last month, was widely exp over from current chair Fresco, later this year. But

## Computing $\pi^{T}$

## A Big Problem

Solve $\boldsymbol{\pi}^{T}=\boldsymbol{\pi}^{T} \mathbf{P}$

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\begin{aligned}
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& \boldsymbol{\pi}^{T}(\mathbf{I}-\mathbf{P})=0
\end{aligned}
$$

## CLEVE'S CORNER <br> THE WORLD'S LARGEST MATRIX COMPUTATION

## Google's PageRank is an eigenvector of a matrix of order 2.7 billion.

One of the reasons why Google is such an effective search engine is the PageRank ${ }^{\text {™ }}$ algorithm, developed by Google's founders, Larry Page and Sergey Brin, when they were graduate students at Stanford University. PageRank is determined entirely by the link structure of the Web. It is recomputed about once a month and does not involve any of the actual content of Web pages or of any individual query. Then, for any particular query, Google finds the pages on the Web that match that query and lists those pages in the order of their PageRank.

Imagine surfing the Web, going from page to page by randomly choosing an outgoing link from one page to get to the next. This can lead to dead ends at pages with no outgoing links, or cycles around cliques of interconnected pages. So, a certain fraction of the time, simply choose a random page from anywhere on the Web. This theoretical random walk of the Web is a Markov chain or Markov process. The limiting probability that a dedicated random surfer visits any particular page is its PageRank. A page has high rank if it has links to and from other pages with high rank.

Let $W$ be the set of Web pages that can reached by following a chain of hyperlinks starting from a page at Google and let $n$ be the number of pages in $W$. The set $W$ actually varies with time, but in May 2002, $n$ was about 2.7 billion. Let $G$ be the $n$-by- $n$ connectivity matrix of

## BY CLEVE MOLER

It tells us that the largest eigenvalue of $A$ is equal to one and that the corresponding eigenvector, which satisfies the equation

$$
x=A x,
$$

exists and is unique to within a scaling factor. When this scaling factor is chosen so that

$$
\sum_{i} x_{i}=1
$$

then $x$ is the state vector of the Markov chain. The elements of $x$ are Google's PageRank.

If the matrix were small enough to fit in MATLAB, one way to compute the eigenvector $x$ would be to start with a good approximate solution, such as the PageRanks from the previous month, and simply repeat the assignment statement

$$
x=A x
$$

until successive vectors agree to within specified tolerance. This is known as the power method and is about the only possible approach for very large $n$. I'm not sure how Google actually computes PageRank, but one step of the power method would require one pass over a database of Web pages, updating weighted reference counts generated by the hyperlinks between pages.

## Computing $\pi^{T}$

## A Big Problem

Solve $\boldsymbol{\pi}^{T}=\boldsymbol{\pi}^{T} \mathbf{P}$
(eigenvector problem)
$\pi^{T}(\mathbf{I}-\mathbf{P})=0$
(too big for direct solves)
Start with $\boldsymbol{\pi}_{0}^{T}=\mathbf{e} / n$ and iterate $\boldsymbol{\pi}_{j+1}^{T}=\boldsymbol{\pi}_{j}^{T} \mathbf{P} \quad$ (power method)

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Measured in days

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A Bigger Problem - Updating
Pages \& links are added, deleted, changed continuously

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Google says just start from scratch every 3 to 4 weeks

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Convergence Time
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## A Bigger Problem - Updating

Pages \& links are added, deleted, changed continuously
Google says just start from scratch every 3 to 4 weeks
Prior results don't help to restart

## Perron Complementation

## Perron Frobenius

$$
\mathbf{P} \geq \mathbf{0}, \text { irreducible } \quad \Longrightarrow \quad \rho(\mathbf{P})=\rho \in \sigma(\mathbf{P}) \quad \text { (simple) }
$$

## Perron Complementation

## Perron Frobenius

$\mathbf{P} \geq \mathbf{0}$, irreducible $\quad \Longrightarrow \quad \rho(\mathbf{P})=\rho \in \sigma(\mathbf{P}) \quad$ (simple)
Unique Left-Hand Perron Vector

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\boldsymbol{\pi}^{T} \mathbf{P}=\rho \boldsymbol{\pi}^{T} \quad \boldsymbol{\pi}^{T}>\mathbf{0} \quad\left\|\boldsymbol{\pi}^{T}\right\|_{1}=\mathbf{1}
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Partition \& Aggregate $\quad \mathbf{P}=\left[\begin{array}{ll}\mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22}\end{array}\right]$

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Shift P by $\rho \longrightarrow$ Schur Complements $\longrightarrow$ Shift back by $\rho$
Perron Complements

$$
\mathbf{S}_{1}=\mathbf{P}_{11}+\mathbf{P}_{12}\left(\rho \mathbf{I}-\mathbf{P}_{22}\right)^{-1} \mathbf{P}_{21} \quad \mathbf{S}_{2}=\mathbf{P}_{22}+\mathbf{P}_{21}\left(\rho \mathbf{I}-\mathbf{P}_{11}\right)^{-1} \mathbf{P}_{12}
$$

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$$

Inherited Properties

$$
\mathbf{S}_{i} \geq \mathbf{0}
$$

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$\mathbf{S}_{i}$ is irreducible

$$
\rho\left(\mathbf{S}_{i}\right)=\rho=\rho(\mathbf{P})
$$

## Exact Aggregation

## Aggregation Matrix

$$
\begin{aligned}
& \mathbf{s}_{i}^{T}=\text { Left-hand Perron vector for } \mathbf{S}_{i} \\
& \mathbf{A}=\left[\begin{array}{ll}
\mathbf{s}_{1}^{T} \mathbf{S}_{1} \mathbf{e} & \mathbf{s}_{1}^{T} \mathbf{S}_{2} \mathbf{e} \\
\mathbf{s}_{2}^{T} \mathbf{S}_{1} \mathbf{e} & \mathbf{s}_{2}^{T} \mathbf{S}_{2} \mathbf{e}
\end{array}\right]_{2 \times 2}
\end{aligned}
$$

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\rho(\mathbf{A})=\rho=\rho(\mathbf{P})=\rho\left(\mathbf{S}_{i}\right)
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\end{aligned}
$$

Inherited Properties

$$
\mathbf{A} \geq 0
$$

A is irreducible

$$
\rho(\mathbf{A})=\rho=\rho(\mathbf{P})=\rho\left(\mathbf{S}_{i}\right)
$$

The Aggregation/Disaggregation Theorem
Left-hand Perron vector for $\mathbf{A}=\left(\alpha_{1}, \alpha_{2}\right)$
Left-hand Perron vector for $\mathbf{P}=\left(\alpha_{1} \mathbf{s}_{1}^{T} \mid \alpha_{2} \mathbf{s}_{2}^{T}\right)$

## Updating By Aggregation

## Prior Data

$\mathbf{Q}_{m \times m}=$ Old Google Matrix

$$
\phi^{T}=\left(\phi_{1}, \phi_{2}, \ldots, \phi_{m}\right)=\text { Old PageRank Vector }
$$

## Updating By Aggregation

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$$
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& \mathbf{Q}_{m \times m}=\text { Old Google Matrix } \\
& \phi^{T}=\left(\phi_{1}, \phi_{2}, \ldots, \phi_{m}\right)=\text { Old PageRank Vector }
\end{aligned}
$$

## Updated Data

$$
\begin{aligned}
& \mathbf{P}_{n \times n}=\text { New Google Matrix } \\
& \boldsymbol{\pi}^{T}=\left(\pi_{1}, \pi_{2}, \ldots, \pi_{n}\right)=\text { New PageRank Vector }
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\end{aligned}
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Separate Pages Likely To Be Most Affected

$$
G=\{\text { most affected }\} \quad \bar{G}=\{\text { less affected }\} \quad \mathcal{S}=G \cup \bar{G}
$$

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Separate Pages Likely To Be Most Affected

$$
G=\{\text { most affected }\} \quad \bar{G}=\{\text { less affected }\} \quad \mathcal{S}=G \cup \bar{G}
$$

New pages (and neighbors) go into $G$

## Aggregation

## Partitioned Matrix

$$
\begin{gathered}
\mathbf{P}_{n \times n}=\begin{array}{cc}
G & \bar{G} \\
G \\
G
\end{array}\left(\begin{array}{cc}
\mathbf{P}_{11} & \mathbf{P}_{12} \\
\mathbf{P}_{21} & \mathbf{P}_{22}
\end{array}\right)=\left[\begin{array}{c|c|c|c}
p_{11} & \cdots & p_{1 g} & \mathbf{r}_{1}^{T} \\
\hline \vdots & \ddots & \vdots & \vdots \\
\hline p_{g 1} & \cdots & p_{g g} & \mathbf{r}_{g}^{T} \\
\hline \mathbf{c}_{1} & \cdots & \mathbf{c}_{g} & \mathbf{P}_{22}
\end{array}\right] \\
\boldsymbol{\pi}^{T}=\left(\pi_{1}, \ldots \pi_{g} \mid \pi_{g+1}, \ldots, \pi_{n}\right)
\end{gathered}
$$

## Aggregation

## Partitioned Matrix

$$
\begin{gathered}
\left.\mathbf{P}_{n \times n}=\begin{array}{cc}
G & \bar{G} \\
G \\
\bar{G} & \mathbf{P}_{11} \\
\mathbf{P}_{12} \\
\mathbf{P}_{21} & \mathbf{P}_{22}
\end{array}\right)=\left[\begin{array}{c|c|c|c}
p_{11} & \cdots & p_{1 g} & \mathbf{r}_{1}^{T} \\
\hline \vdots & \ddots & \vdots & \vdots \\
\hline p_{g 1} & \cdots & p_{g g} & \mathbf{r}_{g}^{T} \\
\mathbf{c}_{1} & \cdots & \mathbf{c}_{g} & \mathbf{P}_{22}
\end{array}\right] \\
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Perron Complements

$$
\begin{aligned}
p_{11} \cdots p_{g g} \text { are } 1 \times 1 & \Longrightarrow \text { Perron complements }=1 \\
& \Longrightarrow \text { Perron vectors } \quad=1
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Perron Complements
$p_{11} \cdots p_{g g}$ are $1 \times 1 \Longrightarrow$ Perron complements $=1$
$\Longrightarrow$ Perron vectors $=1$
One significant complement $\mathbf{S}_{2}=\mathbf{P}_{22}+\mathbf{P}_{21}\left(\mathbf{I}-\mathbf{P}_{11}\right)^{-1} \mathbf{P}_{12}$
One significant Perron vector $\quad \mathbf{s}_{2}^{T} \mathbf{S}_{2}=\mathbf{s}_{2}^{T}$
$\mathrm{A} / \mathrm{D}$ Theorem $\Longrightarrow \mathbf{s}_{2}^{T}=\left(\pi_{g+1}, \ldots, \pi_{n}\right) / \sum_{i=g+1}^{n} \pi_{i}$

## Approximate Aggregation

Use Some Old PageRanks to Approximate New Ones

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\left(\pi_{g+1}, \ldots, \pi_{n}\right) \approx\left(\phi_{g+1}, \ldots, \phi_{n}\right)
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Approximate Aggregation Matrix

$$
\tilde{\mathbf{A}}=\left[\begin{array}{cc}
\mathbf{P}_{11} & \mathbf{P}_{12} \mathbf{e} \\
\widetilde{\mathrm{~s}}_{2}^{T} \mathbf{P}_{21} & 1-\widetilde{\mathrm{s}}_{2}^{T} \mathbf{P}_{21} \mathbf{e}
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Approximate New PageRank Vector

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Improve By Successive Aggregation / Disaggregation?

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Use $\tilde{\widetilde{\pi}}^{T}$ as input to another approximate aggregation step

## Convergence

## THEOREM

Always converges to the new PageRank vector $\boldsymbol{\pi}^{T}$

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Find a relatively small $G$ to minimize $\left|\lambda_{\mathbf{2}}\left(\mathbf{S}_{\mathbf{2}}\right)\right|$

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Can do - Use "power law" distribution of the web

## Conclusions

Elegant Blend of NA, LA, Graph Theory, MC, \& CS

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## Thanks For Your Attention

