



Updating Markov Chains

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Intro

Assumptions

Very large irreducible chain

- $m = O(10^9)$ number of states
- $\mathbf{Q}_{m \times m}$ old transition matrix
- $\phi^T = (\phi_1, \phi_2, \dots, \phi_m)$ old stationary distribution

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Updates

Change some transition probabilities

Add or delete some states

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Aim

Use ϕ^T to compute π^T

Exact (Theoretical) Updating

Perturbation Formula

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Not Practical For Large Problems

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A Little Better, But Not Great

Censoring

Partition (not necessarily NCD)

$$\mathbf{P}_{n \times n} = \begin{pmatrix} G_1 & G_2 & \cdots & G_k \\ G_1 & \mathbf{P}_{11} & \mathbf{P}_{12} & \cdots & \mathbf{P}_{1k} \\ G_2 & \mathbf{P}_{21} & \mathbf{P}_{22} & \cdots & \mathbf{P}_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G_k & \mathbf{P}_{k1} & \mathbf{P}_{k2} & \cdots & \mathbf{P}_{kk} \end{pmatrix}$$

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Visits to states outside of G_i are ignored.

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$$\mathbf{C}_i = \mathbf{P}_{ii} + \mathbf{P}_{i\star}(\mathbf{I} - \mathbf{P}_i^*)^{-1}\mathbf{P}_{\star i}$$

Stochastic Complements

Aggregation

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$$\mathbf{s}_i^T \mathbf{C}_i = \mathbf{s}_i^T$$

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Induced Matrix Partition

$$\mathbf{P}_{n \times n} = \frac{G}{\overline{G}} \begin{pmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{pmatrix} = \begin{bmatrix} p_{11} & \cdots & p_{1g} & \mathbf{P}_{1\star} \\ \vdots & & \vdots & \vdots \\ p_{g1} & \cdots & p_{gg} & \mathbf{P}_{g\star} \\ \mathbf{P}_{\star 1} & \cdots & \mathbf{P}_{\star g} & \mathbf{P}_{22} \end{bmatrix}$$

Specialized Aggregation

Censored Transition Matrices

$$\mathbf{C}_1 = \cdots = \mathbf{C}_g = [1]_{1 \times 1} \quad \mathbf{C}_{g+1} = \mathbf{P}_{22} + \mathbf{P}_{21}(\mathbf{I} - \mathbf{P}_{11})^{-1}\mathbf{P}_{12}$$

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$$\pi^T = (\pi_1, \dots, \pi_g \mid \bar{\pi}^T) = (\alpha_1, \dots, \alpha_g, \mid \alpha_{g+1} \mathbf{s}_{g+1}^T)$$

Old Distribution

$$\phi^T = (\phi_1, \phi_2, \dots \mid \bar{\phi}^T)$$

The Assumption

$$\bar{\phi}^T \approx \bar{\pi}^T \implies \bar{\phi}^T \approx \alpha_{g+1} \mathbf{s}_{g+1}^T \implies \mathbf{s}_{g+1}^T \approx \frac{\bar{\phi}^T}{\bar{\phi}^T \mathbf{e}}$$

Specialized Aggregation

Censored Transition Matrices

$$\mathbf{C}_1 = \dots = \mathbf{C}_g = [1]_{1 \times 1} \quad \mathbf{C}_{g+1} = \mathbf{P}_{22} + \mathbf{P}_{21}(\mathbf{I} - \mathbf{P}_{11})^{-1}\mathbf{P}_{12}$$

Censored Distributions

$$\mathbf{s}_1^T = \dots = \mathbf{s}_g^T = 1 \quad \mathbf{s}_{g+1}^T = \mathbf{s}_{g+1}^T \mathbf{C}_{g+1}$$

Aggregation Matrix

$$\mathcal{A} = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12}\mathbf{e} \\ \mathbf{s}_{g+1}^T \mathbf{P}_{21} & 1 - \mathbf{s}_{g+1}^T \mathbf{P}_{21}\mathbf{e} \end{bmatrix}_{(g+1) \times (g+1)} \longrightarrow \alpha^T = (\alpha_1, \dots, \alpha_g, \alpha_{g+1})$$

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Old Distribution

$$\phi^T = (\phi_1, \phi_2, \dots \mid \bar{\phi}^T)$$

(reordered)

The Assumption

$$\bar{\phi}^T \approx \bar{\pi}^T \implies \bar{\phi}^T \approx \alpha_{g+1} \mathbf{s}_{g+1}^T \implies \mathbf{s}_{g+1}^T \approx \frac{\bar{\phi}^T}{\bar{\phi}^T \mathbf{e}}$$

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Aggregation Theorem

Old Distribution

$$\phi^T = (\phi_1, \phi_2, \dots | \overline{\phi}^T)$$

The Assumption

$$\overline{\phi}^T \approx \overline{\pi}^T \implies \overline{\phi}^T \approx \alpha_{g+1} \mathbf{s}_{g+1}^T \implies \mathbf{s}_{g+1}^T \approx \frac{\overline{\phi}^T}{\overline{\phi}^T \mathbf{e}}$$

Aggregated Distribution

$$\pi^T = (\alpha_1, \dots, \alpha_g, | \alpha_{g+1} \mathbf{s}_{g+1}^T)$$

Updated Distribution

Summary

Reorder & Partition Updated State Space

$$\mathcal{S} = G \cup \overline{G}$$

$G = \{\text{New + Most affected}\}$ $\overline{G} = \{\text{Less affected}\}$

Summary

Reorder & Partition Updated State Space

$$\mathcal{S} = G \cup \overline{G}$$

G = {New + Most affected} \overline{G} = {Less affected}

Use Less Affected Components From Old Distribution

$\overline{\phi}^T \leftarrow$ components from ϕ^T corresponding to states in \overline{G}

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$$\pi^T \leftarrow (\alpha_1, \dots, \alpha_g, |\alpha_{g+1} \mathbf{s}^T|)$$

Iterate ?

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Reorder & Partition Updated State Space

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$G = \{\text{New + Most affected}\}$ $\overline{G} = \{\text{Less affected}\}$

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$\overline{\phi}^T \leftarrow$ components from ϕ^T corresponding to states in \overline{G}

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$$\mathcal{A} \leftarrow \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12}\mathbf{e} \\ \mathbf{s}^T \mathbf{P}_{21} & 1 - \mathbf{s}^T \mathbf{P}_{21}\mathbf{e} \end{bmatrix}_{(g+1) \times (g+1)}$$

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$$\alpha^T \leftarrow (\alpha_1, \dots, \alpha_g, \alpha_{g+1})$$

Approximate Updated Distribution

$$\pi^T \leftarrow (\alpha_1, \dots, \alpha_g, |\alpha_{g+1} \mathbf{s}^T|)$$

Iterate ?

No! — At A Fixed Point

Iterative Aggregation

Move Off Of Fixed Point With Power Step

$$\mathbf{s}^T \leftarrow \overline{\phi}^T / (\overline{\phi}^T \mathbf{e})$$

$$\rightarrow \mathcal{A} \leftarrow \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12}\mathbf{e} \\ \mathbf{s}^T \mathbf{P}_{21} & 1 - \mathbf{s}^T \mathbf{P}_{21} \mathbf{e} \end{bmatrix}_{(g+1) \times (g+1)}$$

$$\boldsymbol{\alpha}^T \leftarrow (\alpha_1, \dots, \alpha_g, \alpha_{g+1})$$

$$\boldsymbol{\pi}^T \leftarrow (\alpha_1, \dots, \alpha_g, | \alpha_{g+1} \mathbf{s}^T)$$

$$\boldsymbol{\psi}^T \leftarrow \boldsymbol{\pi}^T \mathbf{P}$$

(Also makes progress toward convergence when aperiodic)

If $\|\boldsymbol{\psi}^T - \boldsymbol{\chi}^T\| < \tau$ then quit — else

$$\mathbf{s}^T \leftarrow \overline{\boldsymbol{\psi}}^T / (\overline{\boldsymbol{\psi}}^T \mathbf{e})$$

Iterative Aggregation

Move Off Of Fixed Point With Power Step

$$\mathbf{s}^T \leftarrow \overline{\phi}^T / (\overline{\phi}^T \mathbf{e})$$

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Theorem

If $\mathbf{C} = \mathbf{P}_{22} + \mathbf{P}_{21}(\mathbf{I} - \mathbf{P}_{11})^{-1}\mathbf{P}_{12}$ is aperiodic, then convergent for all partitions $\mathcal{S} = G \cup \overline{G}$

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— $|\lambda_2(\mathbf{C})|$ determines rate of convergence

Google's PageRank

Random Walk On WWW Link Structure

$$H_{ij} = \begin{cases} 1/(\text{total } \# \text{ outlinks from page } \mathcal{P}_i) & \text{if } \mathcal{P}_i \rightarrow \mathcal{P}_j, \\ 0 & \text{otherwise} \end{cases}$$

Google's PageRank

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Google Matrix

$$\mathbf{P} = \alpha(\mathbf{H} + \mathbf{E}) + (1 - \alpha)\mathbf{F}$$

- $(\mathbf{H} + \mathbf{E})$ & \mathbf{F} are stochastic $rank(\mathbf{E}) = rank(\mathbf{F}) = 1$
- $0 < \alpha < 1$

Google's PageRank

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- PageRank = π^T

Google's PageRank

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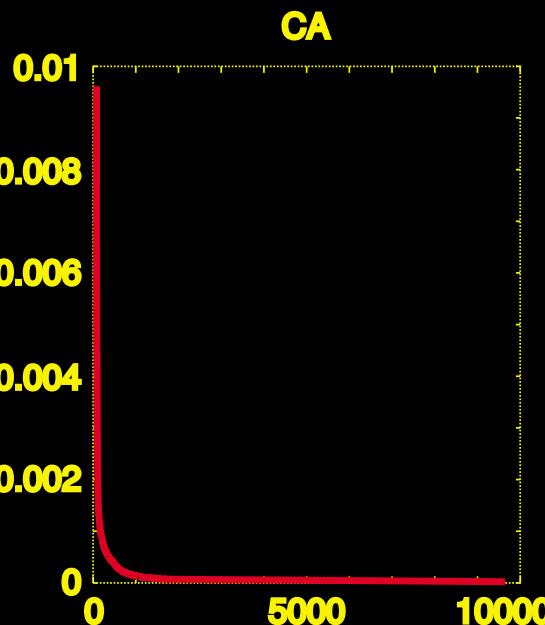
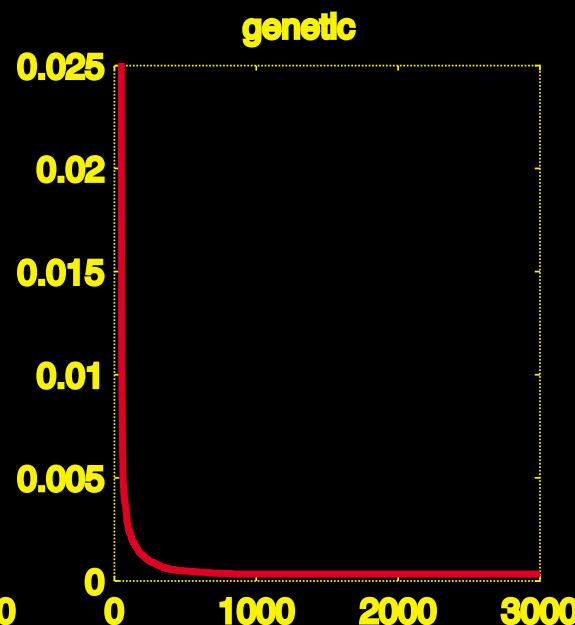
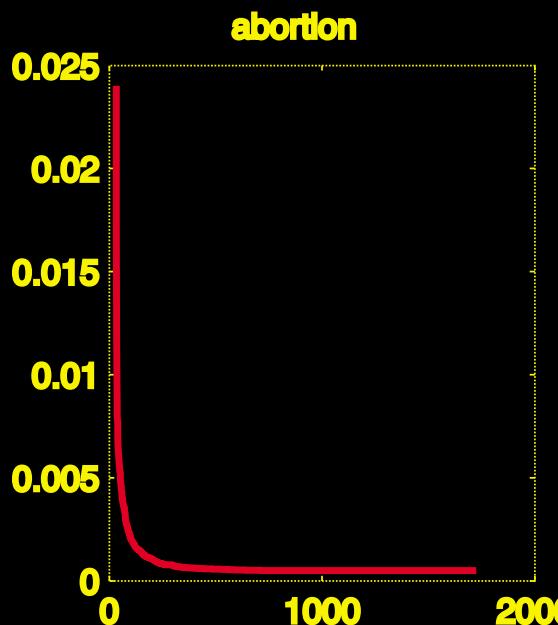
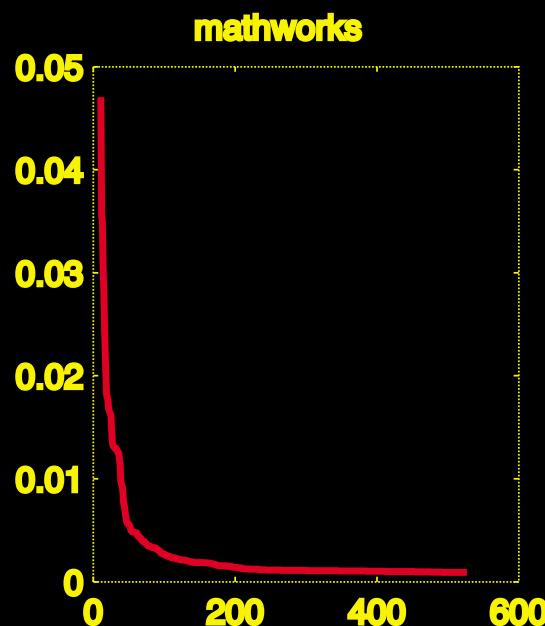
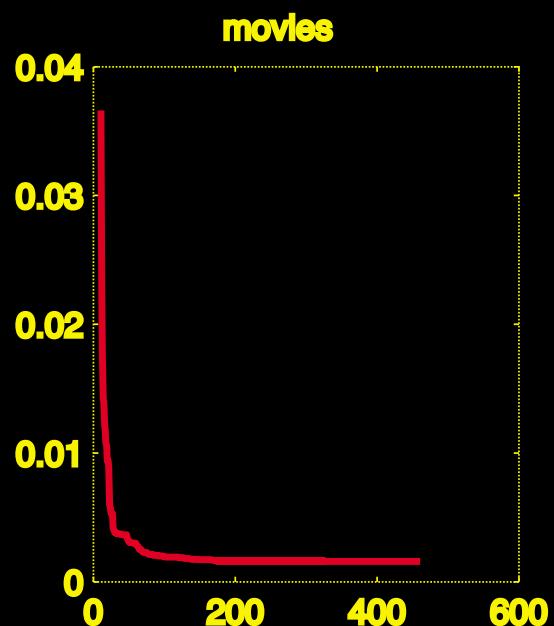
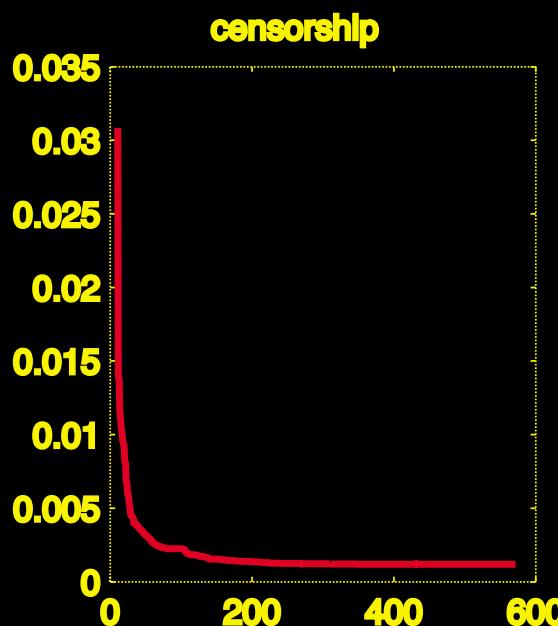
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- $0 < \alpha < 1$
- PageRank = π^T

Power Law Distribution

If ordered by magnitude $\pi(1) \geq \pi(2) \geq \dots \geq \pi(n)$, then

- $\pi(i) \approx \alpha i^{-k}$ for $k \approx 2.109$ [Donato, Laura, Leonardi, 2002]
[Pandurangan, Raghavan, & Upfal, 2004]
- Relatively few large states (i.e., important sites)

“L” Curves



Experiments

The Updates

Nodes Added = 3

Nodes Removed = 50

Links Added = 10

(Different values have little effect on results)

Links Removed = 20

Stopping Criterion

1-norm of residual $< 10^{-10}$

Movies

Power Method

Iterations	Time
------------	------

17	.40
----	-----

Iterative Aggregation

$ G $	Iterations	Time
-------	------------	------

5	12	.39
---	----	-----

10	12	.37
----	----	-----

15	11	.36
----	----	-----

20	11	.35
----	----	-----

25	11	.31
----	----	-----

50	9	.31
----	---	-----

100	9	.33
-----	---	-----

200	8	.35
-----	---	-----

300	7	.39
-----	---	-----

400	6	.47
-----	---	-----

nodes = 451 links = 713

Censorship

Power Method

Iterations	Time
------------	------

38	1.40
----	------

Iterative Aggregation

$ G $	Iterations	Time
-------	------------	------

5	38	1.68
---	----	------

10	38	1.66
----	----	------

15	38	1.56
----	----	------

20	20	1.06
----	----	------

25	20	1.05
----	----	------

50	10	.69
----	----	-----

100	8	.55
-----	---	-----

200	6	.53
-----	---	-----

300	6	.65
-----	---	-----

400	5	.70
-----	---	-----

nodes = 562 links = 736

MathWorks

Power Method

Iterations	Time
------------	------

54	1.25
----	------

Iterative Aggregation

$ G $	Iterations	Time
-------	------------	------

5	53	1.18
---	----	------

10	52	1.29
----	----	------

15	52	1.23
----	----	------

20	42	1.05
----	----	------

25	20	1.13
----	----	------

50	18	.70
----	----	-----

100	16	.70
-----	----	-----

200	13	.70
-----	----	-----

300	11	.83
-----	----	-----

400	10	1.01
-----	----	------

nodes = 517 links = 13,531

Abortion

Power Method

Iterations	Time
------------	------

106	37.08
-----	-------

Iterative Aggregation

$ G $	Iterations	Time
-------	------------	------

5	109	38.56
10	105	36.02
15	107	38.05
20	107	38.45
25	97	34.81
50	53	18.80
100	13	5.18
250	12	5.62
500	6	5.21
750	5	10.22
1000	5	14.61

nodes = 1,693 links = 4,325

Genetics

Power Method

Iterations	Time
------------	------

92	91.78
----	-------

Iterative Aggregation

$ G $	Iterations	Time
-------	------------	------

5	91	88.22
10	92	92.12
20	71	72.53
50	25	25.42
100	19	20.72
250	13	14.97
500	7	11.14
1000	5	17.76
1500	5	31.84

nodes = 2,952 links = 6,485

California

Power Method

Iterations	Time
------------	------

176	5.85
-----	------

Iterative Aggregation

$ G $	Iterations	Time
-------	------------	------

500	19	1.12
-----	----	------

1000	15	.92
------	----	-----

1250	20	1.04
------	----	------

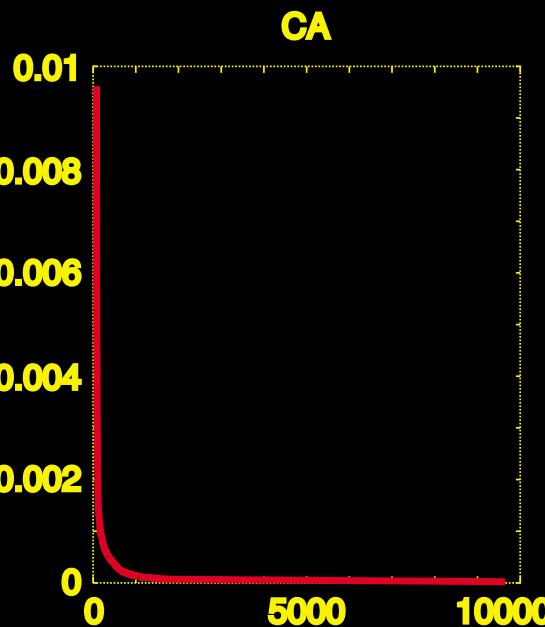
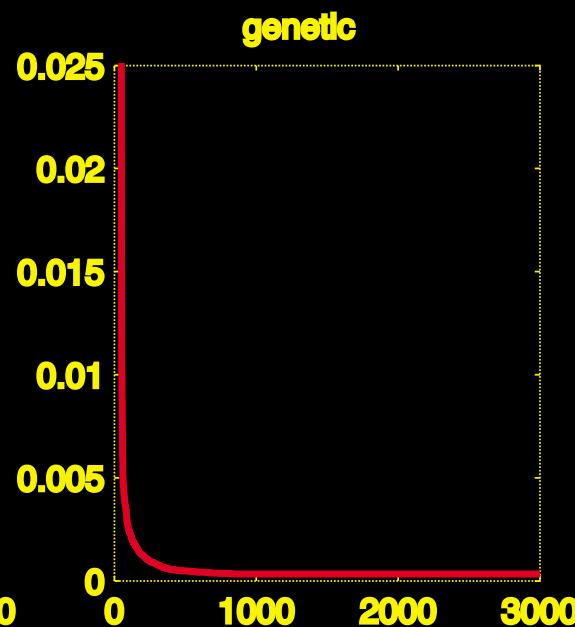
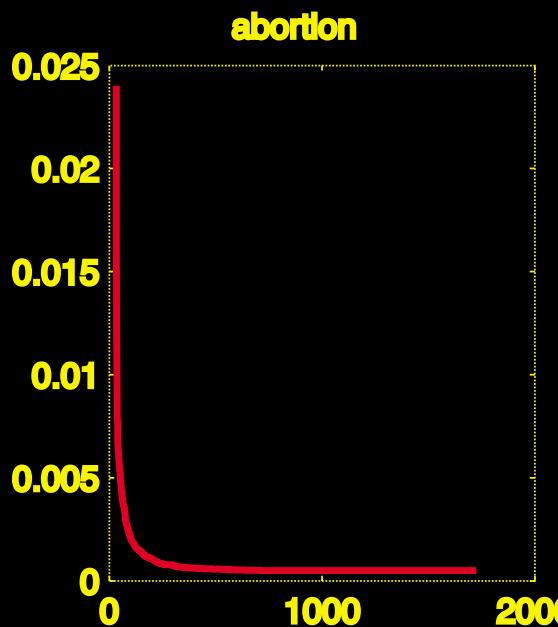
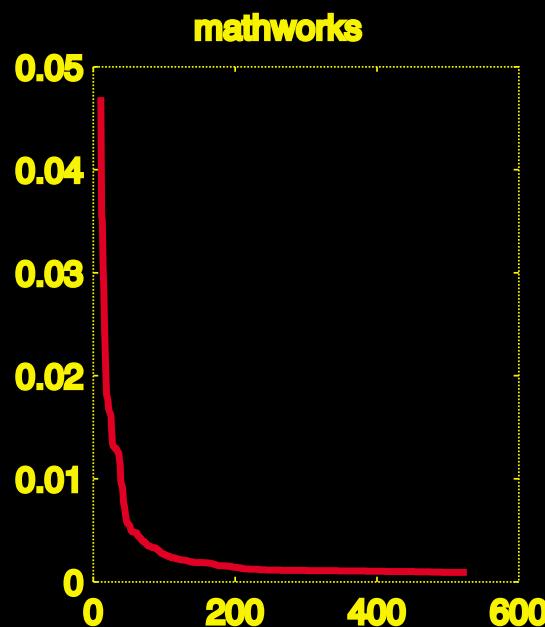
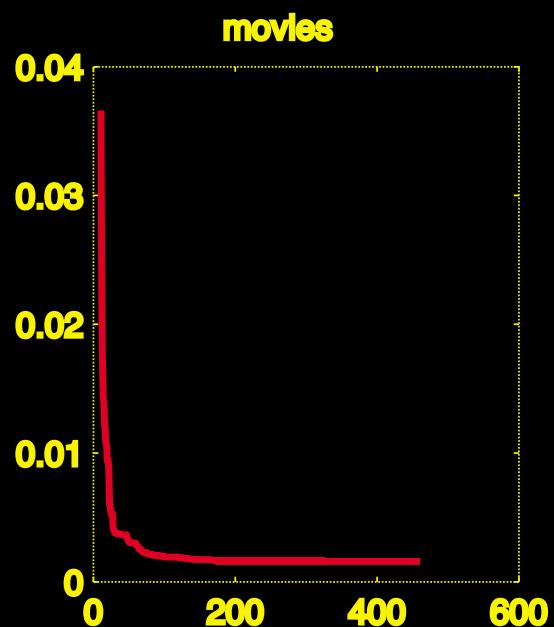
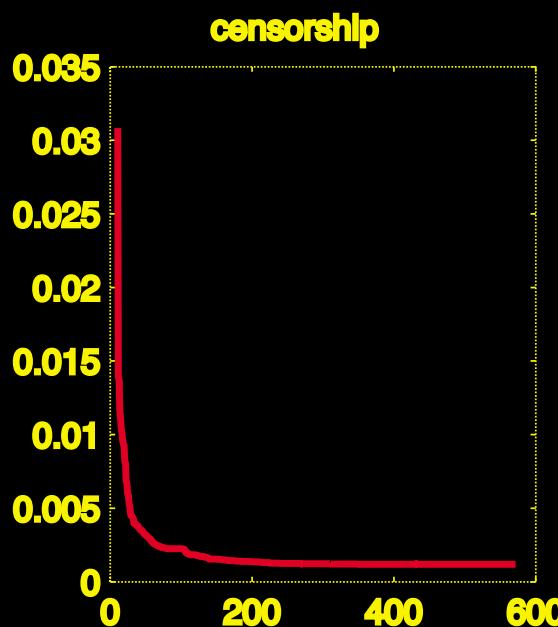
1500	14	.90
------	----	-----

2000	13	1.17
------	----	------

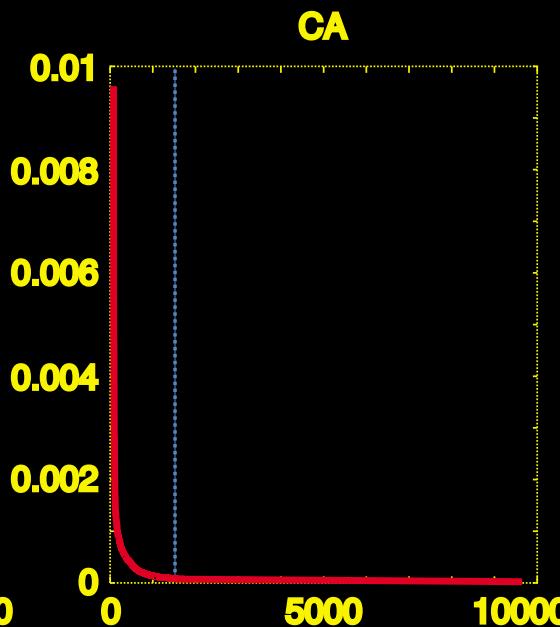
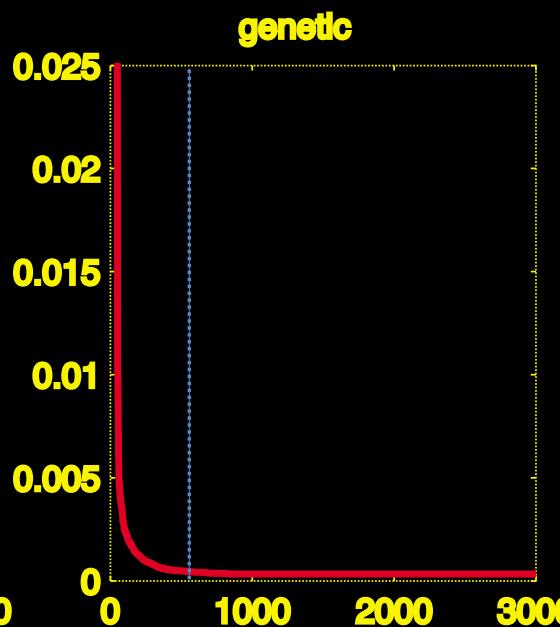
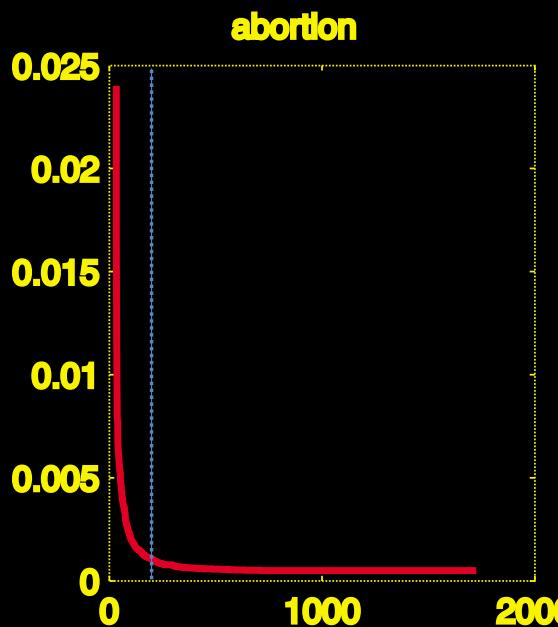
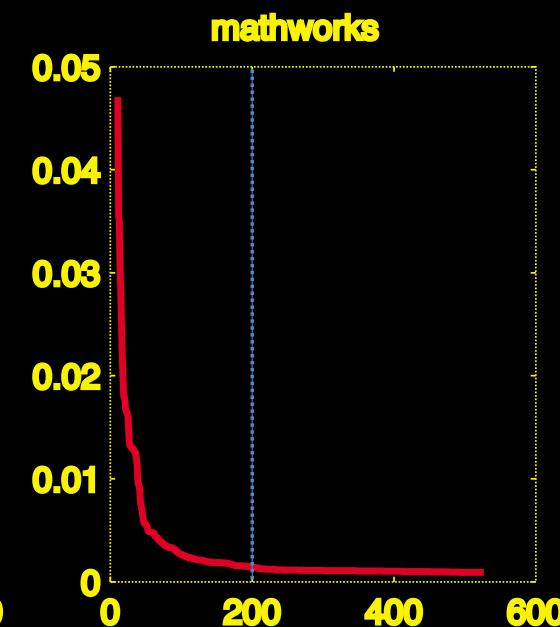
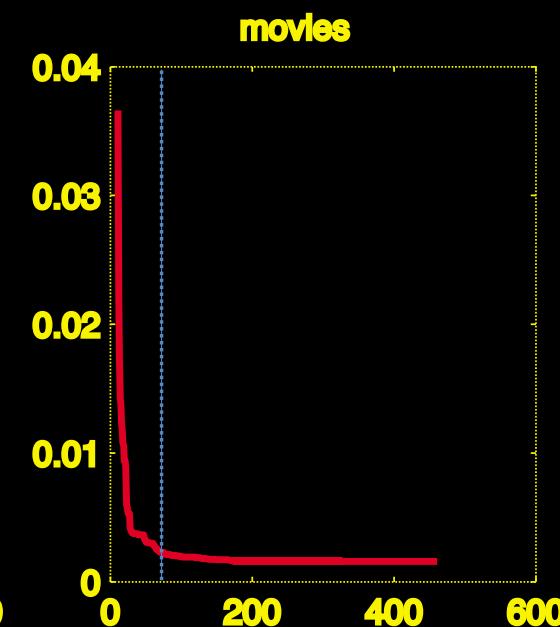
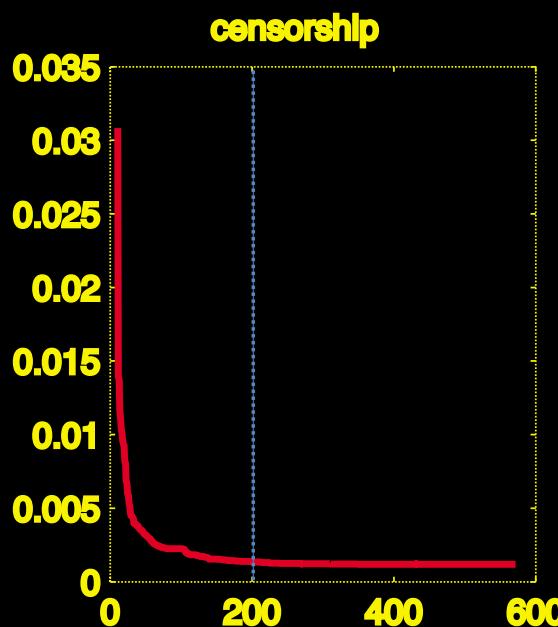
5000	6	1.25
------	---	------

nodes = 9,664 links = 16,150

“L” Curves



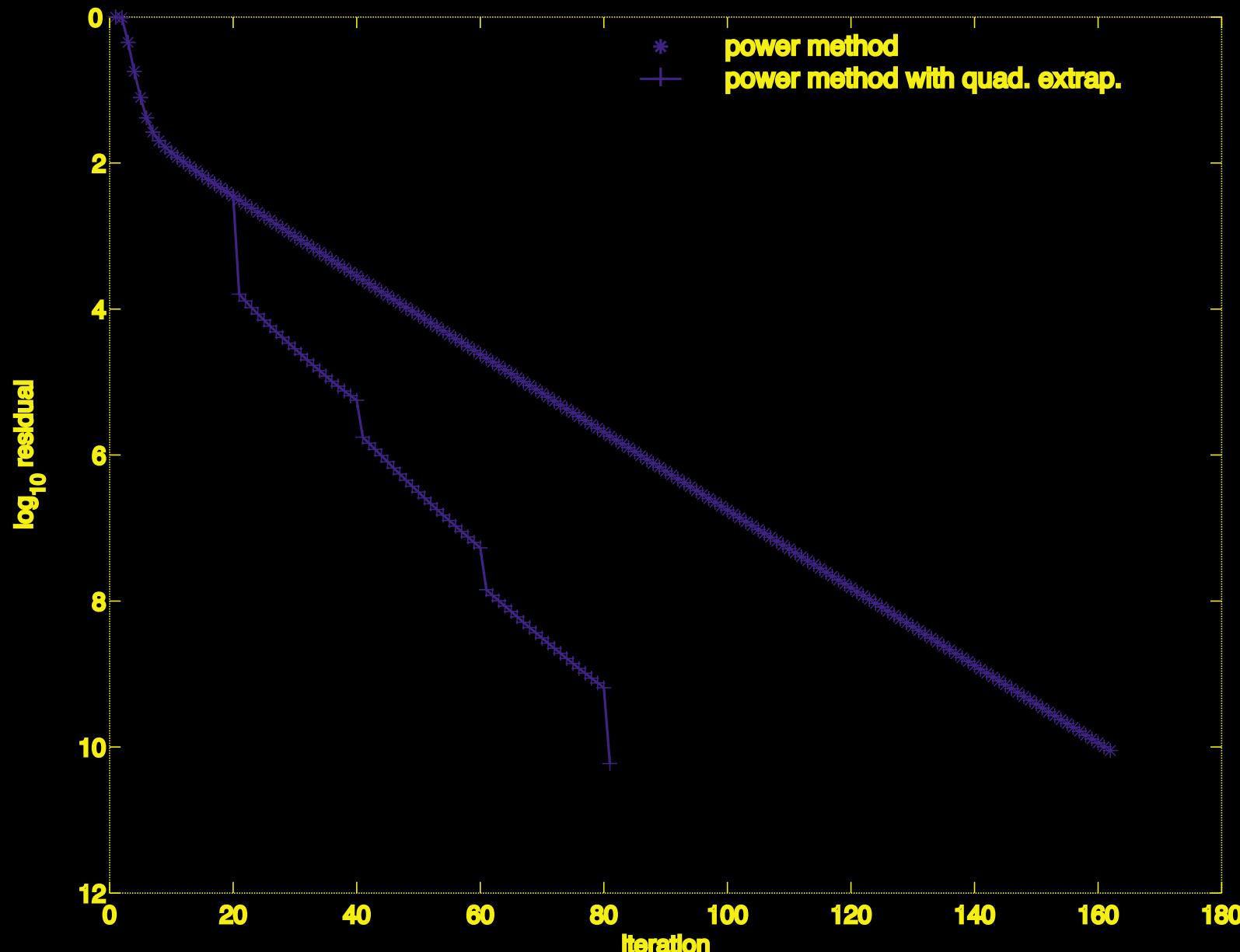
“L” Curves



Quadratic Extrapolation

nodes = 10,000 links = 101,118

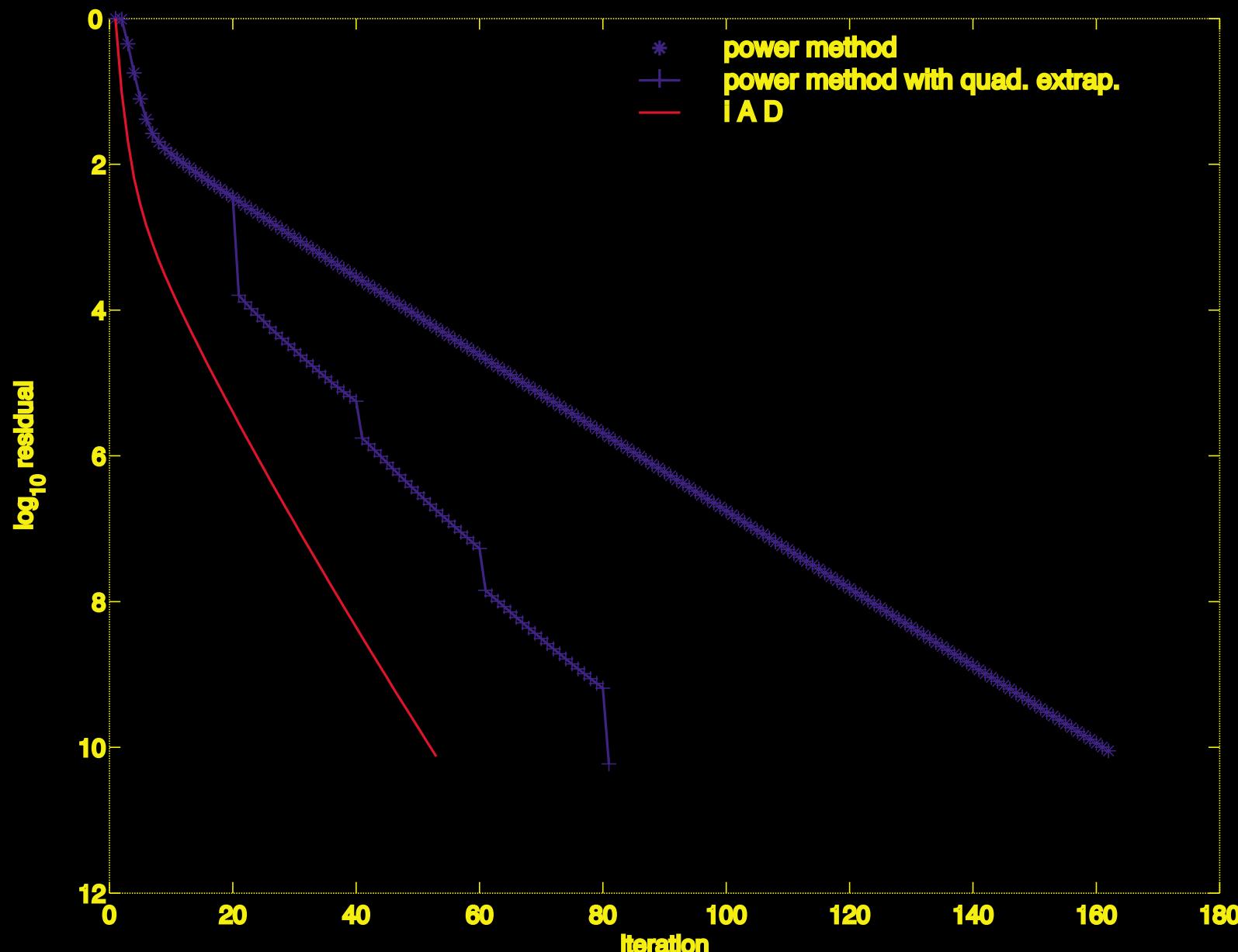
[Kamvar, Haveliwala, Manning, Golub, 2003]



Quadratic Extrapolation

nodes = 10,000 links = 101,118

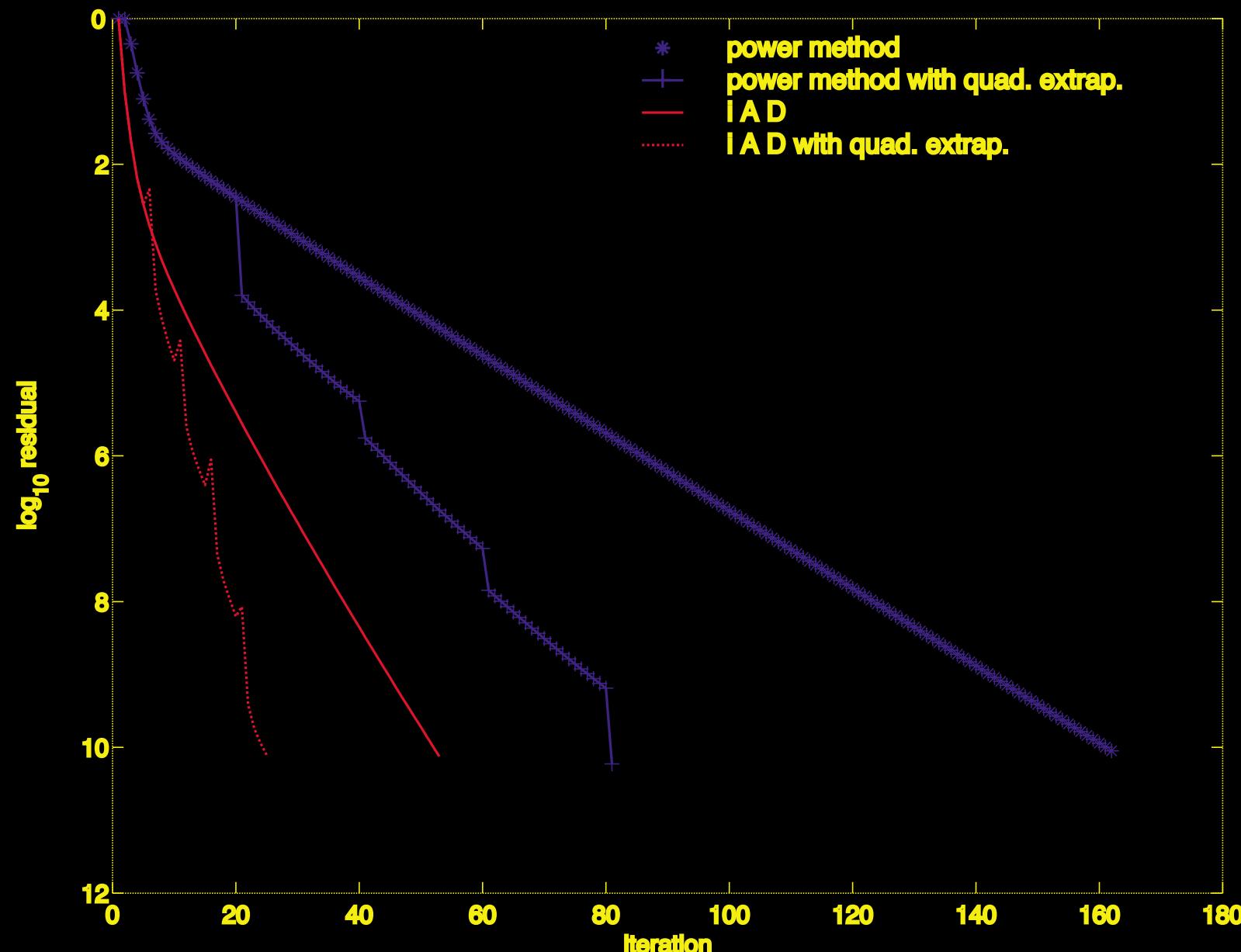
[Kamvar, Haveliwala, Manning, Golub, 2003]



Quadratic Extrapolation

nodes = 10,000 links = 101,118

[Kamvar, Haveliwala, Manning, Golub, 2003]



Timings

	Iterations	Time (sec)	$ G $
Power	162	9.69	
Power+Quad	81	5.93	
IAD	21	2.22	2000
IAD+Quad	16	1.85	2000

nodes = 10,000 links = 101,118

Conclusion

- ★ Iterative aggregation shows promise for updating Markov chains
- ★ Especially for those having power law distributions

Leveling Off Point

$$\pi(i) \approx \alpha i^{-k}$$

$$\left| \frac{d\pi(i)}{di} \right| \approx \epsilon \quad \text{for some user-defined tolerance } \epsilon$$

$$i_{level} \approx \left(\frac{k\alpha}{\epsilon} \right)^{1/k+1}$$

Perhaps better:

$$i_{level} \approx f(n) \left(\frac{k\alpha}{\epsilon} \right)^{1/k+1}$$

For WWW:

$$g_{opt} \approx f(n) \left[\frac{2.109\alpha}{\epsilon} \right]^{1/3.109}$$