LINEAR ALGEBRA AND ITS APPLICATIONS

ELSEVIER Linear Algebra and its Applications 335 (2001) 137–150

www.elsevier.com/locate/laa

# Comparison of perturbation bounds for the stationary distribution of a Markov chain

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Received 18 November 1999; accepted 22 November 2000

Submitted by V. Mehrmann

**Abstract**

The purpose of this paper is to review and compare the existing perturbation bounds for the stationary distribution of a finite, irreducible, homogeneous Markov chain. © 2001 Elsevier Science Inc. All rights reserved.

*AMS classification:* 65F35; 60J10; 15A51; 15A12; 15A18

*Keywords:* Markov chains; Stationary distribution; Stochastic matrix; Group inversion; Sensitivity analysis; Perturbation theory; Condition numbers

## 1. Introduction

Let $P$ be the transition probability matrix of an $n$ state finite, irreducible, homogeneous Markov chain. The stationary distribution vector of $P$ is the unique positive vector $\pi^{\mathrm{T}}$ satisfying

$$\pi^{\mathrm{T}} P = \pi^{\mathrm{T}}, \qquad \sum_{j=1}^{n} \pi_j = 1.$$

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1 Supported in part by National Science Foundation grants CCR-9731856 and DMS-9714811.



PII: S 0 0 2 4 - 3 7 9 5 ( 0 1 ) 0 0 3 2 0 - 2

Suppose $P$ is perturbed to a matrix $\widetilde{P}$, that is the transition probability matrix of an $n$ state finite, irreducible, homogeneous Markov chain as well. Denoting the stationary distribution vector of $\widetilde{P}$ by $\widetilde{\pi}$, the goal is to describe the change $\pi - \widetilde{\pi}$ in the stationary distribution in terms of the change $E \equiv P - \widetilde{P}$ in the transition probability matrix. For suitable norms,

$$\|\pi - \widetilde{\pi}\| \leqslant \kappa \|E\|$$

for various different condition numbers $\kappa$. We review eight existing condition numbers $\kappa_1, \ldots, \kappa_8$. Most of the condition numbers we consider are expressed in terms of either the fundamental matrix of the underlying Markov chain or the group inverse of $I - P$. The condition number $\kappa_8$ is expressed in terms of *mean first passage times*, providing a qualitative interpretation of error bound. In Section 4, we compare the condition numbers.

## 2. Notation

Throughout the article the matrix $P$ denotes the transition probability matrix of an $n$ state finite, irreducible, homogeneous Markov chain $\mathscr{C}$ and $\pi$ denotes the stationary distribution vector. Then

$$\pi^{\mathrm{T}} P = \pi^{\mathrm{T}}, \qquad \pi > 0, \qquad \pi^{\mathrm{T}} e = 1,$$

where $e$ is the column vector of all ones. The perturbed matrix $\widetilde{P} = P - E$ is the transition probability matrix of another $n$ state finite, irreducible, homogeneous Markov chain $\widetilde{\mathscr{C}}$ with the stationary distribution vector $\widetilde{\pi}$:

$$\widetilde{\pi}^{\mathrm{T}} \widetilde{P} = \widetilde{\pi}^{\mathrm{T}}, \qquad \widetilde{\pi} > 0, \qquad \widetilde{\pi}^{\mathrm{T}} e = 1.$$

The identity matrix of size $n$ is denoted by $I$. For a matrix $B$, the $(i, j)$ component of $B$ is denoted by $b_{ij}$.

The 1-norm $\|v\|_1$ of a vector $v$ is the absolute entry sum, and the $\infty$-norm $\|B\|_\infty$ of a matrix $B$ is its maximum absolute row sum.

## 3. Condition numbers of a Markov chain

The norm-wise perturbation bounds we review in this section are of the following form:

$$\|\pi - \widetilde{\pi}\|_p \leqslant \kappa_l \|E\|_q,$$

where $(p, q) = (\infty, \infty)$ or $(1, \infty)$, depending on $l$.

Most of the perturbation bounds we will consider are in terms of one the two matrices related to the chain $\mathscr{C}$: the *fundamental matrix* and the *group inverse* of $A \equiv I - P$. The *fundamental matrix* of the chain $\mathscr{C}$ is defined by

$$Z \equiv (A + e\pi^{\mathrm{T}})^{-1}.$$

The *group inverse* of $A$ is the unique square matrix $A^{\#}$ satisfying

$$AA^{\#}A = A, \quad A^{\#}AA^{\#} = A^{\#}, \quad AA^{\#} = A^{\#}A.$$

The lists of the condition numbers $\kappa_l$ and the references are as follows:

| | |
|---|---|
| $\kappa_1 = \\|Z\\|_\infty$ | Schweitzer [17] |
| $\kappa_2 = \\|A^{\#}\\|_\infty$ | Meyer [12] |
| $\kappa_3 = \dfrac{\max_j (a^{\#}_{jj} - \min_i a^{\#}_{ij})}{2}$ | Haviv and van Heyden [5] |
| | Kirkland et al. [9] |
| $\kappa_4 = \max_{i,j} \|a^{\#}_{ij}\|$ | Funderlic and Meyer [4] |
| $\kappa_5 = \dfrac{1}{1 - \tau_1(P)}$ | Seneta [21] |
| $\kappa_6 = \tau_1(A^{\#}) \;=\; \tau_1(Z)$ | Seneta [22] |
| $\kappa_7 = \dfrac{\min_j \\|A^{-1}_{(j)}\\|_\infty}{2}$ | Ipsen and Meyer [6] |
| | Kirkland et al. [9] |
| $\kappa_8 = \frac{1}{2} \max_j \left[ \dfrac{\max_{i \neq j} m_{ij}}{m_{jj}} \right]$ | Cho and Meyer [1] |

where $m_{ij}, \ i \neq j$, is the *mean first passage time* from state $i$ to state $j$, and $m_{jj}$ is the the *mean return time* state $j$.

### 3.1. Schweitzer [17]

Kemeny and Snell [8] call $Z$ the fundamental matrix of the chain because most of the questions concerning the chain can be answered in terms of $Z$. For instance, the stationary distribution vector of the perturbed matrix $\widetilde{P}$ can be expressed in terms of $\pi$ and $Z$:

$$\widetilde{\pi}^{\mathrm{T}} = \pi^{\mathrm{T}}(I + EZ)^{-1} \quad \text{and} \quad \pi^{\mathrm{T}} - \widetilde{\pi}^{\mathrm{T}} = \widetilde{\pi}^{\mathrm{T}}EZ. \tag{3.1}$$

Eq. (3.1), by Schweitzer [17], gives the first perturbation bound:

$$\|\pi - \widetilde{\pi}\|_1 \leqslant \|Z\|_\infty \|E\|_\infty.$$

We define

$$\kappa_1 \equiv \|Z\|_\infty.$$

### 3.2. *Meyer [12]*

The second matrix related to $\mathscr{C}$ is the group inverse of $A$. In his papers [11,12], Meyer showed that the group inverse $A^\#$ can be used in a similar way $Z$ is used, and, since

$$Z = A^\# + e\pi^{\mathrm{T}},$$

'*all relevant information is contained in* $A^\#$, *and the term* $e\pi^{\mathrm{T}}$ *is redundant*' [14]. In fact, in the place of (3.1), we have

$$\tilde{\pi}^{\mathrm{T}} = \pi^{\mathrm{T}}(I + EA^\#)^{-1} \quad \text{and} \quad \pi^{\mathrm{T}} - \tilde{\pi}^{\mathrm{T}} = \tilde{\pi}^{\mathrm{T}} EA^\#, \tag{3.2}$$

and the resulting perturbation bound is [12]

$$\|\pi - \tilde{\pi}\|_1 \leqslant \|A^\#\|_\infty \, \|E\|_\infty.$$

We define the second condition number

$$\kappa_2 \equiv \|A^\#\|_\infty.$$

### 3.3. *Haviv and van Heyden [5] and Kirkland et al. [9]*

The perturbation bound in this section is derived from (3.2) with a use of the following lemma:

**Lemma 3.1.** *For any vector $d$ and for any vector $c$ such that $c^{\mathrm{T}}e = 0$,*

$$|c^{\mathrm{T}}d| \leqslant \|c\|_1 \, \frac{\max_{i,j} \; |d_i - d_j|}{2}.$$

The resulting perturbation bound is

$$\|\pi - \tilde{\pi}\|_\infty \leqslant \frac{\max_j \, (a^\#_{jj} - \min_i \; a^\#_{ij})}{2} \, \|E\|_\infty.$$

(cf. Refs. [5,9].) We define

$$\kappa_3 \equiv \frac{\max_j \, (a^\#_{jj} - \min_i \; a^\#_{ij})}{2}.$$

### *3.4. Funderlic and Meyer [4]*

Eq. (3.2) provides a component-wise bound for the stationary distribution vector:

$$|\pi_j - \widetilde{\pi}_j| \leqslant \max_i \ |a^{\#}_{ij}| \ \|E\|_\infty,$$

which leads us to

$$\|\pi - \widetilde{\pi}\|_\infty \leqslant \max_{i,j} \ |a^{\#}_{ij}| \ \|E\|_\infty.$$

Funderlic and Meyer [4] called the number

$$\kappa_4 \equiv \max_{i,j} \ |a^{\#}_{ij}|$$

the *chain condition number*.

The behaviour of $\kappa_4$ is provided by Meyer [13,14]. He showed that the size of $\kappa_4$ is primarily governed by how close the subdominant eigenvalues of the chain are to 1. (This is not true for arbitrary matrices. For an example, see [14, p.716].) To be precise, denoting the eigenvalues of $P$ by $1, \lambda_2, \ldots, \lambda_n$, the lower bound and the upper bound of $\kappa_4$ are given by

$$\frac{1}{n \ \min_i |1 - \lambda_i|} \leqslant \max_{i,j} \ |a^{\#}_{ij}| < \frac{2(n-1)}{\prod_i (1 - \lambda_i)}. \tag{3.3}$$

Hence, if the chain is well-conditioned, then all subdominant eigenvalues must be well separated from 1, and if all subdominant eigenvalues are well separated from 1, then the chain must be well-conditioned.

In [13,14], it is indicated that the upper bound $2(n-1)/\prod_i (1-\lambda_i)$ in (3.3) is a rather conservative estimate of $\kappa_4$. If no single eigenvalue of $P$ is extremely close to 1, but enough eigenvalues are within range of 1, then $2(n-1)/\prod_i (1-\lambda_i)$ is large, even if the chain is not too badly condition. Seneta [23] provides a condition number and its bounds to overcome this problem. (See Section 3.6.)

### *3.5. Seneta [21]*

The condition numbers $\kappa_1$, $\kappa_2$, and $\kappa_4$ are in terms of matrix norms. Seneta [21], however, proposed the *ergodicity coefficient* instead of the matrix norm. The *ergodicity coefficient* $\tau_1(B)$ of a matrix $B$ with equal row sums $b$ is defined by

$$\tau_1(B) \equiv \sup_{\substack{\|v\|_1 = 1 \\ v^{\mathrm{T}} e = 0}} \|v^{\mathrm{T}} B\|_1.$$

Note that $\tau_1(B)$ is an ordinary norm of $B$ on the hyperspace $\mathbf{H}^n \equiv \{v : v \in \mathbb{R}^n,\ v^{\mathrm{T}}e = 0\}$ of $\mathbb{R}^n$. eigenvalues are to $b$. For more discussion and study, we refer readers to [2,3,7,10,15,16,18–23,25,26].

For the stochastic matrix $P$, the ergodicity coefficient satisfies $0 \leqslant \tau_1(P) \leqslant 1$. In case of $\tau_1(P) < 1$, we have a perturbation bound in terms of the ergodicity coefficient of $P$ [21]:

$$\|\pi - \widetilde{\pi}\|_1 \leqslant \frac{1}{1-\tau_1(P)}\|E\|_\infty. \tag{3.4}$$

(For the case $\tau_1(P) = 1$, see [22].) We denote

$$\kappa_5 \equiv \frac{1}{1-\tau_1(P)}.$$

### 3.6. *Seneta [22]*

In the previous parts we noted that the group inverse $A^\#$ of $A = I - P$ can be used in place of Kemeny and Snell's fundamental matrix $Z$. In fact, if we use ergodicity coefficients as a measure of sensitivity of the stationary distribution, then $Z$ and $A^\#$ give exactly the same information:

$$\kappa_6 \equiv \tau_1(A^\#) = \tau_1(Z),$$

which is the condition number in the perturbation bound given by Seneta [22]:

$$\|\pi - \widetilde{\pi}\|_1 \leqslant \tau_1(A^\#)\,\|E\|_\infty \ \ (= \tau_1(Z)\,\|E\|_\infty).$$

In Section 3.4, we observed that the size of the condition number $\kappa_4 = \max_{i,j}\ |a^\#_{ij}|$ is governed by the closeness of the subdominant eigenvalues of $P$ to 1, giving the lower and upper bound for $\kappa_4$. However, the problem of overestimating the upper bound for $\kappa_4$ occurs if enough eigenvalues of $P$ are within the range of 1, even if no single eigenvalue $\lambda_i$ of $P$ is close to 1. The following bounds for $\kappa_5$ overcome this problem:

$$\frac{1}{\min_i |1-\lambda_i|} \leqslant \tau_1(A^\#) \leqslant \sum_i \frac{1}{1-\lambda_i} \leqslant \frac{n}{\min_i |1-\lambda_i|}.$$

Unlike the upper bound (3.3) for $\kappa_4$, the far left upper bound for $\kappa_6$ takes only the closest eigenvalue to 1 into account. Hence, it shows that as long as the closest eigenvalue of $P$ is not close to 1, the chain is well-conditioned.

### 3.7. *Ipsen and Meyer [6], and Kirkland et al. [9]*

Ipsen and Meyer [6] derived a set of perturbation bounds and showed that all stationary probabilities react in a uniform manner to perturbations in the transition probabilities. The main result of their paper is based on the following component-wise perturbation bounds:

$$\begin{aligned} \left|\frac{\pi_j - \tilde{\pi}_j}{\pi_j}\right| &\leqslant \|A_{(j)}^{-1}\|_\infty \, \|E\|_\infty, && \text{for all } j = 1, 2, \dots, n, \\ |\pi_k - \tilde{\pi}_k| &\leqslant \min_j \, \|A_{(j)}^{-1}\|_\infty \, \|E\|_\infty, && \text{for all } k = 1, 2, \dots, n, \end{aligned} \tag{3.5}$$

where $A_{(j)}$ is the principal submatrix of $A$ obtained by deleting the $j$th row and column from $A$.

Kirkland et al. [9] improved the perturbation bound in (3.5) by a factor of 2:

$$\|\pi - \tilde{\pi}\|_\infty < \frac{\min_j \, \|A_{(j)}^{-1}\|}{2} \|E\|_\infty,$$

giving the condition number

$$\kappa_7 \equiv \frac{\min_j \, \|A_{(j)}^{-1}\|_\infty}{2}.$$

### 3.8. *Cho and Meyer [1]*

In the previous sections, we saw a number of perturbation bounds for the stationary distribution vector of an irreducible Markov chain. Unfortunately, they provide little qualitative information about the sensitivity of the underlying Markov chain. Moreover, the actual computation of the corresponding condition number is usually expensive relative to computation of the stationary distribution vector itself.

In this section a perturbation bound is presented in terms of the structure of the underlying Markov chain. To be more precise, the condition number $\kappa$ is in terms of *mean first passage times*.

For an $n$-state irreducible Markov chain $\mathscr{C}$, the *mean first passage time* $m_{ij}$ from state $i$ to state $j$ ($j \neq i$) is defined to be the expected number of steps to enter in state $j$ for the first time, starting in state $i$. The *mean return time* $m_{jj}$ of state $j$ is the expected number of steps to return to state $j$ for the first time, starting in state $j$.

Using the relationship

$$m_{jj} = \frac{1}{\pi_j} \quad \text{and} \quad a_{ij}^{\#} = a_{jj}^{\#} - \pi_j m_{ij}, \quad i \neq j,$$

we obtain the perturbation bound

$$\|\pi - \tilde{\pi}\|_\infty \leqslant \frac{1}{2} \max_j \left[ \frac{\max_{i \neq j} m_{ij}}{m_{jj}} \right] \|E\|_\infty$$

(cf. [1]). We define

$$\kappa_8 \equiv \frac{1}{2} \max_j \left[ \frac{\max_{i \neq j} m_{ij}}{m_{jj}} \right].$$

Viewing sensitivity in terms of mean first passage times can sometimes help practitioners decide whether or not to expect sensitivity in their Markov chain models merely by observing the structure of the chain without computing or estimating condition numbers. For example, consider chains consisting of a dominant central state with strong connections to and from all other states. Physical systems of this type have historically been called *mammillary systems* (see [24,27]). The simplest example of a mammillary Markov chain is one whose transition probability matrix has the form

$$P = \begin{pmatrix} 1-p_1 & 0 & \cdots & 0 & p_1 \\ 0 & 1-p_2 & \cdots & 0 & p_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1-p_k & p_k \\ q_1 & q_2 & \cdots & q_k & 1-\sum_j q_j \end{pmatrix} \tag{3.6}$$

in which the $p_i$'s and $q_i$'s are not unduly small (say, $p_i > .5$ and $q_i \approx 1/k$). Mean first passage times $m_{ij}$, $i \neq j$, in mammillary structures are never very large. For example, in the simple mammillary chain defined by (3.6) it can be demonstrated that

$$m_{ij} = \begin{cases} \frac{1}{p_i} & \text{when } j = k+1, \\ \frac{1+\sigma}{q_j} - \frac{1}{p_j} & \text{when } i = k+1, \\ \frac{1+\sigma}{q_j} + \frac{1}{p_i} - \frac{1}{p_j} & \text{when } i, j \neq k+1, \end{cases} \quad \text{where} \quad \sigma = \sum_{h=1}^{k} \frac{q_h}{p_h}.$$

Since each $m_{jj} \geqslant 1$, it is clear that $\kappa_8$ cannot be large, and consequently no stationary probability can be unduly sensitive to perturbations in $P$. It is apparent that similar remarks hold for more general mammillary structures as well.

## 4. Comparison of condition numbers

In this section, we compare the condition numbers $\kappa_l$. The norm-wise perturbation bounds in Section 3 are of the following form:

$$\|\pi - \tilde{\pi}\|_p \leqslant \kappa_l \|E\|_q,$$

where $(p, q) = (\infty, \infty)$, or $(1, \infty)$, depending on $l$. (For $\kappa_4$, a strict inequality holds.) The purpose of this section is simply to compare the condition numbers appearing in those norm-wise bounds. Therefore we use the form

$$\frac{\|\pi - \widetilde{\pi}\|_\infty}{\|E\|_\infty} \leqslant \kappa_l$$

for all condition numbers although some of the bounds are tighter in this form. (See Remark 4.1.)

**Lemma 4.1.**

(a) $\max_j \max_{i,k} |a^{\#}_{ij} - a^{\#}_{kj}| = \max_j (a^{\#}_{jj} - \min_i a^{\#}_{ij})$,

(b) $\dfrac{\max_j (a^{\#}_{jj} - \min_i a^{\#}_{ij})}{2} \leqslant \max\limits_{i,j} |a^{\#}_{ij}| < \max\limits_j (a^{\#}_{jj} - \min\limits_i a^{\#}_{ij})$,

(c) $\max_{i,j} |a^{\#}_{ij}| \leqslant \|A^{\#}\|_\infty$,

(d) $\max_j (a^{\#}_{jj} - \min_i a^{\#}_{ij}) \leqslant \tau_1(A^{\#})$,

(e) $\tau_1(A^{\#}) \leqslant n \dfrac{\max\limits_j (a^{\#}_{jj} - \max\limits_i a^{\#}_{ij})}{2}$,

(f) $\tau_1(A^{\#}) \leqslant \|A^{\#}\|_\infty$,
$\tau_1(A^{\#}) \leqslant \|Z\|_\infty$,
$\tau_1(A^{\#}) \leqslant \dfrac{1}{1 - \tau_1(P)}$,

(g) $\|A^{\#}\|_\infty - 1 \leqslant \|Z\|_\infty \leqslant \|A^{\#}\|_\infty + 1$.

**Proof.** (a) Using a symmetric permutation, we may assume that a particular probability occurs in the last position of $\pi$. Partition $A$ and $\pi$ as follows:

$$A = \begin{pmatrix} A_{(n)} & c \\ d^{\mathrm{T}} & a_{nn} \end{pmatrix}, \quad \pi = \begin{pmatrix} \pi_{(n)} \\ \pi_n \end{pmatrix},$$

where $A_{(n)}$ is the principal matrix of $A$ obtained by deleting the $n$th row and column from $A$. Since rank $A = n - 1$, the relationship $a_{nn} = d^{\mathrm{T}} A^{-1}_{(n)} c$ holds, so that

$$A^{\#} = \begin{pmatrix} (I - e\pi^{\mathrm{T}}_{(n)}) A^{-1}_{(n)} (I - e\pi^{\mathrm{T}}_{(n)}) & -\pi_n (I - e\pi^{\mathrm{T}}_{(n)}) A^{-1}_{(n)} e \\ -\pi^{\mathrm{T}}_{(n)} A^{-1}_{(n)} (I - e\pi^{\mathrm{T}}_{(n)}) & \pi_n \pi^{\mathrm{T}}_{(n)} A^{-1}_{(n)} e \end{pmatrix}.$$

Hence

$$a^{\#}_{in} = \begin{cases} -\pi_n e_i^{\mathrm{T}} A^{-1}_{(n)} e + \pi_n \pi^{\mathrm{T}}_{(n)} A^{-1}_{(n)} e, & i \neq n, \\ \pi_n \pi^{\mathrm{T}}_{(n)} A^{-1}_{(n)} e, & i = n, \end{cases} \tag{4.1}$$

where $e_i$ is the $i$th column of $I$. By M-matrix properties, it follows that $A_{(n)}^{-1} > 0$ so that $\pi_n e_i^{\mathrm{T}} A_{(n)}^{-1} e, \pi_n \pi_{(n)}^{\mathrm{T}} A_{(n)}^{-1} e > 0$. Thus

$$\max_i a_{in}^{\#} = a_{nn}^{\#},$$

and the result follows.

(b) For each $j$, $a_{jj}^{\#} > 0$, by (4.1). Since $\pi^{\mathrm{T}} A^{\#} = \mathbf{0}^{\mathrm{T}}$, it follows that there exists $k_0$ such that $a_{k_0 j}^{\#} < 0$. Furthermore, let $i_0$ be such that $\max_i \ |a_{ij}^{\#}| = |a_{i_0 j}^{\#}|$.
Then

$$|a_{i_0 j}^{\#}| < \begin{cases} |a_{i_0 j}^{\#} - a_{k_0 j}^{\#}| & \text{if } a_{i_0 j}^{\#} \geqslant 0, \\ |a_{i_0 j}^{\#} - a_{jj}^{\#}| & \text{otherwise.} \end{cases}$$

Thus, for all $j$,

$$\max_i \ |a_{ij}^{\#}| < \max_{i,k} \ |a_{ij}^{\#} - a_{kj}^{\#}|$$

so that

$$\max_{i,j} |a_{ij}^{\#}| < \max_j \ \max_{i,k} \ |a_{ij}^{\#} - a_{kj}^{\#}|.$$

On the other hand,

$$\frac{\max_j \ \max_{i,k} \ |a_{ij}^{\#} - a_{kj}^{\#}|}{2} \leqslant \frac{\max_{i,j} \ |a_{ij}^{\#}| + \max_{k,j} \ |a_{kj}^{\#}|}{2} = \max_{i,j} |a_{ij}^{\#}|.$$

Now the assertion follows from (a).

(c) It follows directly from the definitions of $1-$, and $\infty-$ norm, and their relationship,

(d) For a real number $a$, define $a^+ \equiv \max\{a, 0\}$. Then

$$\begin{aligned} \max_j \ \max_{i,k} \ |a_{ij}^{\#} - a_{kj}^{\#}| &= \max_{i,k} \ \max_j \ |a_{ij}^{\#} - a_{kj}^{\#}| \\ &\leqslant \max_{i,k} \sum_j (a_{ij}^{\#} - a_{kj}^{\#})^+ \\ &= \tau_1(A^{\#}) \quad \text{(by Seneta [19, p.139])}, \end{aligned}$$

and the assertion follows by (a).

(e) For any vector $x \in \mathbb{R}^n$,

$$\|x\|_1 \leqslant n\|x\|_\infty,$$

and the result follows, since

$$\tau_1(A^\#) = \frac{1}{2} \max_{i,k} \|A^\#_{i*} - A^\#_{k*}\|_1,$$

and

$$\frac{\max_j \ \max_{i,k} \ |a^\#_{ij} - a^\#_{kj}|}{2} = \frac{1}{2} \max_{i,k} \|A^\#_{i*} - A^\#_{k*}\|_\infty,$$

where $B_{i*}$ denotes the $i$th row of a matrix $B$.

(f) Since for any matrix $B$ with equal row sums,

$$\tau_1(B) = \sup_{\substack{\|v\|_1=1 \\ v^{\mathrm{T}}e=0}} \|v^{\mathrm{T}}B\|_1,$$

and

$$\|y^{\mathrm{T}}B\|_1 \leqslant \|y\|_1 \ \|B\|_\infty,$$

for any vector $y \in \mathbb{R}^n$, we have

$$\tau_1(B) \leqslant \|B\|_\infty.$$

The inequalities follow by this relationship together with the following facts, proven in [22]:

$$\tau_1(A^\#) = \tau_1(Z),$$

and if $\tau_1(P) < 1$, then

$$\tau_1(A^\#) \leqslant \frac{1}{1 - \tau_1(P)}.$$

(g) It follows by applying the triangle inequality to

$$Z = A^\# + e\pi^{\mathrm{T}} \quad \text{and} \quad A^\# = Z - e\pi^{\mathrm{T}}. \qquad \square$$

The following list summarizes the relationship between the condition numbers:

*Relation between condition numbers*

$$\kappa_8 = \kappa_3 \leqslant \kappa_4 < 2\kappa_3 \leqslant \kappa_6 \leqslant \kappa_l \qquad \text{for } l = 1, 2, 5,$$
$$\kappa_6 \leqslant n\kappa_3,$$
$$\kappa_2 - 1 \leqslant \kappa_1 \leqslant \kappa_2 + 1.$$

**Remark 4.1.** As remarked at the beginning of this section, some of the condition numbers provide tighter bounds than the form used in this section. To be more precise, for $l = 1, 2, 5, 6$,

$$\frac{\|\pi - \widetilde{\pi}\|_\infty}{\|E\|_\infty} \leqslant \frac{\|\pi - \widetilde{\pi}\|_1}{\|E\|_\infty} \leqslant \kappa_l.$$

To give a 'fairer' comparison to these condition numbers, note that $(\pi - \widetilde{\pi})^{\mathrm{T}} e = 0$ implies $\|\pi - \widetilde{\pi}\|_\infty \leqslant (1/2)\|\pi - \widetilde{\pi}\|_1$ so that

$$\frac{\|\pi - \widetilde{\pi}\|_\infty}{\|E\|_\infty} \leqslant \kappa_l',$$

where $\kappa_l' = (1/2)\kappa_l$, for $l = 1, 2, 5, 6$.[2] The comparison of these 'new' condtion numbers $\kappa_l'$ with $\kappa_3$ is as given above:

$$\kappa_3 \leqslant \kappa_l',$$

for $l = 1, 2, 5, 6$.

## 5. Concluding remarks

We reviewed eight existing perturbation bounds, and the condition numbers are compared. The list at the end of the last section clarifies the relationships between seven condition numbers. Among these seven condition numbers, the smallest condition number is $\kappa_3$

$$\frac{\max_j \,(a^{\#}_{jj} - \min_i \, a^{\#}_{ij})}{2}$$

by Haviv and van Heyde [5] and Kirkland et al. [9] or equivalently, $\kappa_8$

$$\frac{1}{2} \max_j \left[ \frac{\max_{i \neq j} m_{ij}}{m_{jj}} \right]$$

by Cho and Meyer [1].

The only condition number not included in the comparison is

$$\kappa_7 = \frac{\min_j \|A_{(j)}^{-1}\|_\infty}{2}.$$

Is $\kappa_3$ a smaller condition number than $\kappa_7$? Since by (4.1),

$$a^{\#}_{jj} - \min_i a^{\#}_{ij} = \pi_j \|A_{(j)}^{-1}\|_\infty,$$

[2] The authors thank the referee for bringing this point to our attention.

for each $j$, the question is whether

$$\max_j \pi_j \|A_{(j)}^{-1}\|_\infty \leqslant \min_j \|A_{(j)}^{-1}\|_\infty \tag{5.1}$$

holds. In other words, is $\pi_j$ always small enough so that $\max_j \pi_j \|A_{(j)}^{-1}\|_\infty$ is never larger than $\min_j \|A_{(j)}^{-1}\|_\infty$? This question leads us to the relationship between a stationary probability $\pi_j$ and the corresponding $A_{(j)}^{-1}$.

In a special case, the relationship is clear.

**Lemma 5.1.** *If $P$ is of rank 1, then*

$$\|A_{(j)}^{-1}\|_\infty = \frac{1}{\pi_j}, \qquad \textit{for all } j = 1, 2, \ldots, n.$$

The proof appears in [9, Observation 3.3].

This relationship for an arbitrary irreducible Markov chain does not hold. In general, however, $\|A_{(j)}^{-1}\|_\infty$ tends to increase as $\pi_j$ decreases, and vice versa.

Notice that $\kappa_8$ is equal to $\kappa_3$. However, viewing sensitivity in terms of mean first passage times can sometimes help practitioner decide whether or not to expect sensitivity in their Markov chain models merely by observing the structure of the chain, thus obviating the need for computing or estimating condition numbers.

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