# Uncoupling the Perron Eigenvector Problem*

Carl D. Meyer
*Mathematics Department and*
*Center for Research in Scientific Computation*
*Box 8205*
*North Carolina State University*
*Raleigh, North Carolina 27695-8205*

Dedicated to Alan J. Hoffman on the occasion of his 65th birthday.

Submitted by Charles R. Johnson

ABSTRACT

For a nonnegative irreducible matrix $\mathbf{A}$ with spectral radius $\rho$, this paper is concerned with the determination of the unique normalized Perron vector $\boldsymbol{\pi}$ which satisfies $\mathbf{A}\boldsymbol{\pi} = \rho\boldsymbol{\pi}$, $\boldsymbol{\pi} > 0$, $\sum_j \pi_j = 1$. It is explained how to uncouple a large matrix $\mathbf{A}$ into two or more smaller matrices—say $\mathbf{P}_{11}, \mathbf{P}_{22}, \ldots, \mathbf{P}_{kk}$—such that this sequence of smaller matrices has the following properties: (1) Each $\mathbf{P}_{ii}$ is also nonnegative and irreducible, so that each $\mathbf{P}_{ii}$ has a unique Perron vector $\boldsymbol{\pi}^{(i)}$. (2) Each $\mathbf{P}_{ii}$ has the same spectral radius $\rho$ as $\mathbf{A}$. (3) It is possible to determine the $\boldsymbol{\pi}^{(i)}$'s completely independently of each other, so that one can execute the computation of the $\boldsymbol{\pi}^{(i)}$'s parallel. (4) It is easy to couple the smaller Perron vectors $\boldsymbol{\pi}^{(i)}$ back together in order to produce the Perron vector $\boldsymbol{\pi}$ for the original matrix $\mathbf{A}$.

## 1. INTRODUCTION

For a nonnegative irreducible matrix $\mathbf{A}_{m\times m}$ with spectral radius $\rho(\mathbf{A}) = \rho$, a fundamental problem concerns the determination of the unique normalized Perron vector $\boldsymbol{\pi}_{m\times 1}$ which satisfies

$$\mathbf{A}\boldsymbol{\pi} = \rho\boldsymbol{\pi}, \qquad \boldsymbol{\pi} > 0, \quad \sum_{i=1}^{m} \pi_i = 1.$$

*This work was supported in part by the National Science Foundation under Grant DMS-8521154.

*LINEAR ALGEBRA AND ITS APPLICATIONS* 114/115:69–94 (1989) 69


655 Avenue of the Americas, New York, NY 10010 0024-3795/89/$3.50

For small values of $m$, this is not a difficult problem to solve using adaptations of standard techniques for solving systems of linear equations. However, there exist many applications for which $m$ is too large to be comfortably handled by standard methods.

For large scale problems, it is only natural to attempt to uncouple the original matrix $\mathbf{A}$ somehow into two or more smaller matrices—say $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_k$—of order $r_1, r_2, \dots, r_k$, respectively, where $\sum_{i=1}^{k} r_i = m$. Ideally, this sequence of smaller matrices should have the following properties.

(1) Each smaller matrix $\mathbf{P}_i$ should also be nonnegative and irreducible, so that each $\mathbf{P}_i$ has a unique normalized Perron vector $\pi^{(i)}$.

(2) Each smaller matrix $\mathbf{P}_i$ should have the same spectral radius $\rho$ as the original matrix $\mathbf{A}$.

(3) It should be possible to determine the $\pi^{(i)}$'s completely independently of each other. For modern parallel computer architectures it is desirable to be able to execute the computation of the $\pi^{(i)}$'s in parallel.

(4) Finally, it must be easy to couple the smaller normalized Perron vectors $\pi^{(i)}$ back together in order to produce the normalized Perron vector $\pi$ for the original nonnegative matrix $\mathbf{A}$.

This paper is dedicated to showing how to accomplish the above four goals.

## 2. PERRON COMPLEMENTATION

The purpose of this section is to introduce the concept of a Perron complement in a nonnegative irreducible matrix and to develop some of the basic properties of Perron complementation. The concept of Perron complementation will be the cornerstone for the entire development which follows.

Unless otherwise stated, $\mathbf{A}$ will denote an $m \times m$ nonnegative irreducible matrix with spectral radius $\rho$, and $\mathbf{A}$ will be partitioned as

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1k} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{k1} & \mathbf{A}_{k2} & \cdots & \mathbf{A}_{kk} \end{pmatrix},$$

where all diagonal blocks are square. It is well known that $\rho\mathbf{I}-\mathbf{A}$ is a singular $M$-matrix of rank $m-1$ and that every principal submatrix of $\rho\mathbf{I}-\mathbf{A}$ of order $m-1$ or smaller is a nonsingular $M$-matrix. [See Berman and Plemmons (1979, p. 156)]. In particular, if $\mathbf{A}_i$ denotes the principal submatrix of $\mathbf{A}$ obtained by deleting the $i$th row and $i$th column of blocks from the partitioned form of A, then each $\rho\mathbf{I}-\mathbf{A}_i$ is a nonsingular $M$-matrix. Therefore,

$$(\rho\mathbf{I}-\mathbf{A}_i)^{-1}\geqslant 0, \tag{2.1}$$

so that the following terms are well defined.

DEFINITION 2.1. Let $\mathbf{A}$ be an $m\times m$ nonnegative irreducible[1] matrix with spectral radius $\rho$, and let $\mathbf{A}$ have a $k$-level partition

$$\mathbf{A}=\begin{pmatrix}\mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1k}\\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2k}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{A}_{k1} & \mathbf{A}_{k2} & \cdots & \mathbf{A}_{kk}\end{pmatrix}$$

in which all diagonal blocks are square. For a given index $i$, let $\mathbf{A}_i$ denote the principal block submatrix of $\mathbf{A}$ obtained by deleting the $i$th row and $i$th column of blocks from A, and let $\mathbf{A}_{i*}$ and $\mathbf{A}_{*i}$ designate

$$\mathbf{A}_{i*}=\begin{pmatrix}\mathbf{A}_{i1} & \mathbf{A}_{i2} & \cdots & \mathbf{A}_{i,i-1} & \mathbf{A}_{i,i+1} & \cdots & \mathbf{A}_{ik}\end{pmatrix}$$

and

$$\mathbf{A}_{*i}=\begin{pmatrix}\mathbf{A}_{1i}\\ \vdots\\ \mathbf{A}_{i-1,i}\\ \mathbf{A}_{i+1,i}\\ \vdots\\ \mathbf{A}_{ki}\end{pmatrix}.$$

[1] The irreducibility assumption is not absolutely necessary for this definition to make sense. The $i$th Perron complement is well defined whenever $\rho(\mathbf{A}_i)<\rho(\mathbf{A})$, and many subsequent statements remain valid under these weaker conditions. Of course, irreducibility guarantees that this is the case for all $i$.

That is, $\mathbf{A}_{i*}$ is the $i$th row of blocks with $\mathbf{A}_{ii}$ removed, and $\mathbf{A}_{*i}$ is the $i$th column of blocks with $\mathbf{A}_{ii}$ removed. The *Perron complement* of $\mathbf{A}_{ii}$ in $\mathbf{A}$ is defined to be the matrix

$$\mathbf{P}_{ii} = \mathbf{A}_{ii} + \mathbf{A}_{i*}(\rho\mathbf{I} - \mathbf{A}_i)^{-1}\mathbf{A}_{*i}.$$

For example, the Perron complement of $\mathbf{A}_{22}$ in

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} \\ \mathbf{A}_{31} & \mathbf{A}_{32} & \mathbf{A}_{33} \end{pmatrix}$$

is given by

$$\mathbf{P}_{22} = \mathbf{A}_{22} + \begin{pmatrix} \mathbf{A}_{21} & \mathbf{A}_{23} \end{pmatrix} \begin{pmatrix} \rho\mathbf{I} - \mathbf{A}_{11} & -\mathbf{A}_{13} \\ -\mathbf{A}_{31} & \rho\mathbf{I} - \mathbf{A}_{33} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{A}_{12} \\ \mathbf{A}_{32} \end{pmatrix}.$$

The reason for the terminology *Perron complement* will become clear as later developments unfold. Although the concept of Perron complementation as defined above is not the same as the well-known concept of Schur complementation, there is an obvious connection. For example, consider a 2-level partition of a nonnegative irreducible matrix

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}$$

with spectral radius $\rho$. Assuming square diagonal blocks, the Perron complement of $\mathbf{A}_{11}$ is given by

$$\mathbf{P}_{11} = \mathbf{A}_{11} + \mathbf{A}_{12}(\rho\mathbf{I} - \mathbf{A}_{22})^{-1}\mathbf{A}_{21},$$

and the Perron complement of $\mathbf{A}_{22}$ is

$$\mathbf{P}_{22} = \mathbf{A}_{22} + \mathbf{A}_{21}(\rho\mathbf{I} - \mathbf{A}_{11})^{-1}\mathbf{A}_{12}.$$

For this case, it is easy to see that the Perron complement of $\mathbf{A}_{11}$ is in fact

$$[\rho\mathbf{I} - \mathrm{Schur\,Complement}(\rho\mathbf{I} - \mathbf{A}_{22})] \qquad \text{in the matrix } \rho\mathbf{I} - \mathbf{A}$$

and the Perron complement of $\mathbf{A}_{22}$ is

$$[\rho\mathbf{I} - \mathrm{Schur\,Complement}(\rho\mathbf{I} - \mathbf{A}_{11})] \qquad \text{in the matrix } \rho\mathbf{I} - \mathbf{A}.$$

The following observation—stated as a technical lemma—is conceptually straightforward to prove.

LEMMA 2.1. *Suppose that $\mathbf{A}_{ii}$ has size $r \times r$, and let*

$$\tilde{\mathbf{A}} = \mathbf{QAQ},$$

*where $\mathbf{Q}$ is the elementary permutation matrix which corresponds to an interchange of the 1st and ith block positions. The matrix $\tilde{\mathbf{A}}$ can be repartitioned into a $2 \times 2$ block matrix*

$$\tilde{\mathbf{A}} = \begin{pmatrix} \tilde{\mathbf{A}}_{11} & \tilde{\mathbf{A}}_{12} \\ \tilde{\mathbf{A}}_{21} & \tilde{\mathbf{A}}_{22} \end{pmatrix}$$

*so that $\tilde{\mathbf{A}}_{11} = \mathbf{A}_{ii}$ and the Perron complement of $\mathbf{A}_{ii}$ in $\mathbf{A}$ is the same as the Perron complement of $\tilde{\mathbf{A}}_{11}$ in $\tilde{\mathbf{A}}$. That is,*

$$\mathbf{P}_{ii} = \tilde{\mathbf{P}}_{11} = \tilde{\mathbf{A}}_{11} + \tilde{\mathbf{A}}_{12}(\rho\mathbf{I} - \tilde{\mathbf{A}}_{22})^{-1}\tilde{\mathbf{A}}_{21}.$$

We are now in a position to develop some of the basic properties of Perron complementation. The following theorem is the first in the sequence.

THEOREM 2.1. *If*

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1k} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{k1} & \mathbf{A}_{k2} & \cdots & \mathbf{A}_{kk} \end{pmatrix}$$

*is a nonnegative irreducible matrix with spectral radius* $\rho$, *then each Perron complement*

$$\mathbf{P}_{ii} = \mathbf{A}_{ii} + \mathbf{A}_{i*}(\rho\mathbf{I} - \mathbf{A}_i)^{-1}\mathbf{A}_{*i}$$

*is also a nonnegative matrix whose spectral radius is again given by* $\rho$. *Moreover, if*

$$\boldsymbol{\pi} = \begin{pmatrix} \boldsymbol{\pi}^{(1)} \\ \boldsymbol{\pi}^{(2)} \\ \vdots \\ \boldsymbol{\pi}^{(k)} \end{pmatrix} > \mathbf{0}$$

*is a conformably partitioned positive eigenvector for* $\mathbf{A}$ *associated with* $\rho$, *then*

$$\mathbf{P}_{ii}\boldsymbol{\pi}^{(i)} = \rho\boldsymbol{\pi}^{(i)}. \tag{2.2}$$

*That is,* $\boldsymbol{\pi}^{(i)} > 0$ *is a positive eigenvector for* $\mathbf{P}_{ii}$ *associated with the spectral radius* $\rho$.

*Proof.* Assume that $\mathbf{A}$ has been permuted and repartitioned as described in Lemma 2.1, so that

$$\tilde{\mathbf{A}} = \mathbf{QAQ} = \begin{pmatrix} \tilde{\mathbf{A}}_{11} & \tilde{\mathbf{A}}_{12} \\ \tilde{\mathbf{A}}_{21} & \tilde{\mathbf{A}}_{22} \end{pmatrix}$$

and the Perron complement of $\mathbf{A}_{ii}$ in $\mathbf{A}$ is the same as the Perron complement of $\tilde{\mathbf{A}}_{11}$ in $\tilde{\mathbf{A}}$. That is, $\mathbf{P}_{ii} = \tilde{\mathbf{P}}_{11}$. Since each principal submatrix of $\rho\mathbf{I} - \mathbf{A}$ (and $\rho\mathbf{I} - \tilde{\mathbf{A}}$) of order $m - 1$ or less is a nonsingular $M$-matrix, it follows from (2.1) that $(\rho\mathbf{I} - \tilde{\mathbf{A}}_{22})^{-1} \geqslant 0$ and hence

$$\mathbf{P}_{ii} = \tilde{\mathbf{P}}_{11} = \tilde{\mathbf{A}}_{11} + \tilde{\mathbf{A}}_{12}(\rho\mathbf{I} - \tilde{\mathbf{A}}_{22})^{-1}\tilde{\mathbf{A}}_{21} \geqslant 0. \qquad \blacksquare$$

REMARK. $\mathbf{P}_{ii}$ need not be strictly positive—an example is given after this proof.

To prove that

$$\mathbf{P}_{ii}\pi^{(i)} = \rho\pi^{(i)},$$

define $\tilde{\pi} \equiv \mathbf{Q}\pi$ and write

$$\tilde{\pi} = \mathbf{Q}\pi = \begin{pmatrix} \tilde{\pi}^{(1)} \\ \tilde{\pi}^{(2)} \end{pmatrix},$$

where $\tilde{\pi}^{(1)} = \pi^{(i)}$. Use the facts that $\mathbf{A}\pi = \rho\pi$ and $\mathbf{Q}^2 = \mathbf{I}$ to conclude that

$$\rho\tilde{\pi} = \rho\mathbf{Q}\pi = \mathbf{QA}\pi = \mathbf{QAQ}^2\pi = \tilde{\mathbf{A}}\tilde{\pi}$$

and hence

$$\tilde{\mathbf{A}}_{11}\tilde{\pi}^{(1)} + \tilde{\mathbf{A}}_{12}\tilde{\pi}^{(2)} = \rho\tilde{\pi}^{(1)},$$

$$\tilde{\mathbf{A}}_{21}\tilde{\pi}^{(1)} + \tilde{\mathbf{A}}_{22}\tilde{\pi}^{(2)} = \rho\tilde{\pi}^{(2)}.$$

It now follows that

$$\tilde{\pi}^{(2)} = \left(\rho\mathbf{I} - \tilde{\mathbf{A}}_{22}\right)^{-1}\tilde{\mathbf{A}}_{21}\tilde{\pi}^{(1)},$$

and therefore

$$\begin{aligned}\mathbf{P}_{ii}\pi^{(i)} &= \tilde{\mathbf{P}}_{11}\tilde{\pi}^{(1)} = \tilde{\mathbf{A}}_{11}\tilde{\pi}^{(1)} + \tilde{\mathbf{A}}_{12}\left(\rho\mathbf{I} - \tilde{\mathbf{A}}_{22}\right)^{-1}\tilde{\mathbf{A}}_{21}\tilde{\pi}^{(1)} \\ &= \tilde{\mathbf{A}}_{11}\tilde{\pi}^{(1)} + \tilde{\mathbf{A}}_{12}\tilde{\pi}^{(2)} \\ &= \rho\tilde{\pi}^{(1)} = \rho\pi^{(i)}.\end{aligned}$$

This proves that $\rho$ is an eigenvalue for $\mathbf{P}_{ii}$ with an associated positive eigenvector given by $\pi^{(i)}$. It follows from the classical Perron-Frobenius theory that for a nonnegative irreducible matrix, the only eigenvalue which can have a positive eigenvector is the Perron root. Thus, $\rho$ must indeed be the spectral radius of $\mathbf{P}_{ii}$. ■

REMARK. The method used in the previous proof yields a similar conclusion for any eigenvalue.

To see that a Perron complement need not be strictly positive, consider a $4 \times 4$ nonnegative irreducible matrix whose partitions and zero pattern are shown below:

$$\mathbf{A} = \left(\begin{array}{ccc|c} + & + & 0 & 0 \\ + & + & + & + \\ + & + & + & + \\ \hline + & + & + & + \end{array}\right).$$

For this configuration,

$$\mathbf{P}_{11} = \mathbf{A}_{11} + \mathbf{A}_{12}(\rho\mathbf{I} - \mathbf{A}_{22})^{-1}\mathbf{A}_{21}$$

$$= \begin{pmatrix} + & + & 0 \\ + & + & + \\ + & + & + \end{pmatrix} + \begin{pmatrix} 0 \\ + \\ + \end{pmatrix}[+](+ \quad + \quad +) = \begin{pmatrix} + & + & 0 \\ + & + & + \\ + & + & + \end{pmatrix}.$$

Although Perron complements can have zero entires, the zeros are always in just the right places to guarantee that each $\mathbf{P}_{ii}$ is an irreducible matrix.

THEOREM 2.2. *If*

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1k} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{k1} & \mathbf{A}_{k2} & \cdots & \mathbf{A}_{kk} \end{pmatrix}$$

*is a nonnegative irreducible matrix with spectral radius* $\rho$, *then each Perron complement*

$$\mathbf{P}_{ii} = \mathbf{A}_{ii} + \mathbf{A}_{i*}(\rho\mathbf{I} - \mathbf{A}_i)^{-1}\mathbf{A}_{*i}$$

*is also a nonnegative* irreducible *matrix with spectral radius* $\rho$.

Because they are each of independent interest, two different proofs are given.

*Algebraic proof.* Suppose that $\mathbf{A}_{ii}$—and hence $\mathbf{P}_{ii}$—is $r \times r$. If

$$\boldsymbol{\pi} = \begin{pmatrix} \boldsymbol{\pi}^{(1)} \\ \boldsymbol{\pi}^{(2)} \\ \vdots \\ \boldsymbol{\pi}^{(k)} \end{pmatrix} > \mathbf{0}$$

is a conformably partitioned positive eigenvector for $\mathbf{A}$ associated with its spectral radius $\rho$, then Theorem 2.1 guarantees that $\boldsymbol{\pi}^{(i)} > 0$ is a positive eigenvector for $\mathbf{P}_{ii}$ associated with its spectral radius $\rho$. If $\mathbf{D}$ is the $r \times r$ diagonal matrix

$$\mathbf{D} = \begin{pmatrix} \pi_1^{(i)} & & & \\ & \pi_2^{(i)} & & \\ & & \ddots & \\ & & & \pi_r^{(i)} \end{pmatrix},$$

then

$$\mathbf{S} = \frac{\mathbf{D}^{-1}\mathbf{P}_{ii}\mathbf{D}}{\rho}$$

is a row stochastic matrix. For all row stochastic matrices, it is known that the unit eigenvalue has index 1—i.e., the unit eigenvalue for $\mathbf{S}$ has only linear elementary divisors. [See Gantmacher (1960, Vol. 2, p. 84).] Therefore, as an eigenvalue of $\mathbf{P}_{ii}$, Index$(\rho) = 1$, and hence the Jordan form for $\rho\mathbf{I} - \mathbf{P}_{ii}$ is

$$\mathbf{J}_{\rho\mathbf{I}-\mathbf{P}_{ii}} = \begin{pmatrix} \mathbf{0}_{t\times t} & \mathbf{0} \\ \mathbf{0} & \mathbf{X} \end{pmatrix}_{r \times r}, \tag{2.3}$$

where $t$ is the algebraic multiplicity of $\rho$ as an eigenvalue of $\mathbf{P}_{ii}$ and where $\mathbf{X}$ is a nonsingular. Assume now that $\mathbf{A}$ has been permuted and repartitioned as described in Lemma 2.1, so that

$$\tilde{\mathbf{A}} = \begin{pmatrix} \tilde{\mathbf{A}}_{11} & \tilde{\mathbf{A}}_{12} \\ \tilde{\mathbf{A}}_{21} & \tilde{\mathbf{A}}_{22} \end{pmatrix} \quad \text{and} \quad \mathbf{P}_{ii} = \tilde{\mathbf{P}}_{11}.$$

By observing

$$\begin{pmatrix} \mathbf{I} & \tilde{\mathbf{A}}_{12}(\rho\mathbf{I}-\tilde{\mathbf{A}}_{22})^{-1} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \rho\mathbf{I}-\tilde{\mathbf{A}}_{11} & -\tilde{\mathbf{A}}_{12} \\ -\tilde{\mathbf{A}}_{21} & \rho\mathbf{I}-\tilde{\mathbf{A}}_{22} \end{pmatrix}$$

$$\times \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ (\rho\mathbf{I}-\tilde{\mathbf{A}}_{22})^{-1}\tilde{\mathbf{A}}_{21} & \mathbf{I} \end{pmatrix} = \begin{pmatrix} \rho\mathbf{I}-\tilde{\mathbf{P}}_{11} & \mathbf{0} \\ \mathbf{0} & \rho\mathbf{I}-\tilde{\mathbf{A}}_{22} \end{pmatrix},$$

it follows that

$$\operatorname{Rank}(\rho\mathbf{I}-\tilde{\mathbf{A}}) = \operatorname{Rank}(\rho\mathbf{I}-\tilde{\mathbf{P}}_{11}) + \operatorname{Rank}(\rho\mathbf{I}-\tilde{\mathbf{A}}_{22}).$$

Suppose $\tilde{\mathbf{A}}$, $\tilde{\mathbf{A}}_{11}$, and $\tilde{\mathbf{A}}_{22}$ have dimensions $m \times m$, $r \times r$, and $q \times q$, respectively, with $r + q = m$. It is well known that $\tilde{\mathbf{A}}$ being nonnegative and irreducible implies that $\operatorname{Rank}(\rho\mathbf{I}-\tilde{\mathbf{A}}) = m - 1$ and $\operatorname{Rank}(\rho\mathbf{I}-\tilde{\mathbf{A}}_{22}) = q$. Thus

$$\operatorname{Rank}(\rho\mathbf{I}-\mathbf{P}_{ii}) = \operatorname{Rank}(\rho\mathbf{I}-\tilde{\mathbf{P}}_{11}) = m - 1 - q = r - 1.$$

This together with (2.3) produces the conclusion that $\rho$ must in fact be a simple eigenvalue for $\mathbf{P}_{ii}$. It is not difficult to see that $\mathbf{P}_{ii}^T$ is the Perron complement of $\mathbf{A}_{ii}^T$ in $\mathbf{A}^T$, so that Theorem 2.1 guarantees $\mathbf{P}_{ii}^T$ also has a positive eigenvector associated with its spectral radius $\rho$. It is known (Gantmacher, 1960, Vol. 2, p. 79) that if the spectral radius $\rho$ of a nonnegative matrix is a simple eigenvalue, and if the matrix and its transpose each possess a positive eigenvector associated with $\rho$, then the matrix must in fact be irreducible. Since this is precisely the situation for the Perron complement $\mathbf{P}_{ii}$, the conclusion is that $\mathbf{P}_{ii}$ must be irreducible. ■

The above proof is purely algebraic in nature. However, irreducibility can be viewed strictly as a combinatorial concept because of the fact that a matrix $\mathbf{M}$ is irreducible if and only if its directed graph $\mathscr{G}(\mathbf{M})$ is strongly connected. It is therefore desirable to also argue the irreducibility of a Perron complement from a graph theoretic point of view. The author is indebted to C. R. Johnson for suggesting the following combinatorial proof.

*Combinatorial proof of Theorem* 2.2. For any particular Perron complement $\mathbf{P}_{ii}$ of size $r \times r$, it can be assumed—without loss of generality—that A has been permuted and repartitioned to the form

$$\mathbf{A}_{n\times n}=\begin{pmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{pmatrix},$$

in which $\mathbf{A}_{11}$ is $r\times r$ and $\mathbf{P}_{ii}$ becomes identified with

$$\mathbf{P}_{11}=\mathbf{A}_{11}+\mathbf{A}_{12}(\rho\mathbf{I}-\mathbf{A}_{22})^{-1}\mathbf{A}_{21}.$$

To demonstrate that $\mathscr{G}(\mathbf{P}_{11})$ is strongly connected, let

$$h,k\in\{1,2,\dots,r\},$$

and let $\mathscr{E}(\cdot)$ denote the set of directed edges in the directed graph of a specified matrix.

*Case 1: There is a path from node h to node k in $\mathscr{G}(\mathbf{A}_{11})$.* In this case, there must also be a path from node $h$ to node $k$ in $\mathscr{G}(\mathbf{P}_{11})$, because $\mathscr{E}(\mathbf{A}_{11})\subseteq\mathscr{E}(\mathbf{P}_{11})$.

*Case 2: There is no path from node h to node k in $\mathscr{G}(\mathbf{A}_{11})$.* For this situation, there must be a path from node $h$ to node $k$ in $\mathscr{G}(\mathbf{A})$ but which necessarily passes through at least one node

$$i\in\{r+1,r+2,\dots,n\}.$$

That is, $\mathscr{G}(\mathbf{A})$ contains a sequence of directed edges

$$h\to h_1\cdots\to h_p\to i_1\cdots\to i_q\to k_1\to\cdots\to k_t\to k$$

leading from node $h$ to node $k$ such that

$$\{(h\to h_1),(h_1\to h_2),\dots,(h_{p-1}\to h_p)\}\subseteq\mathscr{E}(\mathbf{A}_{11}), \tag{2.4}$$

$$\{(k_1\to k_2),(k_2\to k_3),\dots,(k_t\to k\}\subseteq\mathscr{E}(\mathbf{A}_{11}), \tag{2.5}$$

and where

$$\varnothing\neq\{i_1,i_2,\dots,i_q\}\subseteq\{r+1,r+2,\dots,n\}.$$

In particular, the existence of directed edges

$$h_p \to i_1 \to \cdots \to i_q \to k_1$$

in $\mathscr{G}(\mathbf{A})$ guarantees that

$$\left[\mathbf{A}_{12}\mathbf{A}_{22}^{q-1}\mathbf{A}_{21}\right]_{h_p k_1} \neq 0.$$

This, together with the fact that

$$\mathbf{A}_{12}(\rho\mathbf{I} - \mathbf{A}_{22})^{-1}\mathbf{A}_{21} = \sum_{j=0}^{\infty} \frac{\mathbf{A}_{12}\mathbf{A}_{22}^{j}\mathbf{A}_{21}}{\rho^{j+1}},$$

implies

$$\left[\mathbf{A}_{12}(\rho\mathbf{I} - \mathbf{A}_{22})^{-1}\mathbf{A}_{21}\right]_{h_p k_1} \neq 0,$$

so that $(h_p \to k_1) \in \mathscr{E}(\mathbf{P}_{11})$. Combining this with (2.4) and (2.5) together with the fact that $\mathscr{E}(\mathbf{A}_{11}) \subseteq \mathscr{E}(\mathbf{P}_{11})$ leads to the desired conclusion that there is a path from node $h$ to node $k$ in $\mathscr{G}(\mathbf{P}_{11})$. ■

## 3. UNCOUPLING AND COUPLING THE PERRON VECTOR

Adopt the following definition—which is consistent with the terminology used in Horn and Johnson (1985).

Definition 3.1. For a nonnegative irreducible matrix $\mathbf{A}$ with spectral radius $\rho$, the unique normalized eigenvector $\boldsymbol{\pi}$ satisfying the conditions

$$\mathbf{A}\boldsymbol{\pi} = \rho\boldsymbol{\pi}, \qquad \boldsymbol{\pi} > 0, \quad \text{and} \quad \mathbf{e}^T\boldsymbol{\pi} = 1,$$

where $\mathbf{e}^T = (1 \;\; 1 \;\; \cdots \;\; 1)$, is called the *Perron vector* for $\mathbf{A}$.

If

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1k} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{k1} & \mathbf{A}_{k2} & \cdots & \mathbf{A}_{kk} \end{pmatrix}$$

is a nonnegative irreducible matrix with spectral radius $\rho$, then the results of the previous section guarantee that each Perron complement

$$\mathbf{P}_{ii} = \mathbf{A}_{ii} + \mathbf{A}_{i*}(\rho\mathbf{I} - \mathbf{A}_i)^{-1}\mathbf{A}_{*i}$$

is also a nonnegative irreducible matrix which again has spectral radius $\rho$. Furthermore, if

$$\boldsymbol{\pi} = \begin{pmatrix} \boldsymbol{\pi}^{(1)} \\ \boldsymbol{\pi}^{(2)} \\ \vdots \\ \boldsymbol{\pi}^{(k)} \end{pmatrix}$$

is the conformably partitioned Perron vector for $\mathbf{A}$, then Theorem 2.1 guarantees that the vector defined by

$$\mathbf{p}_i \equiv \frac{\boldsymbol{\pi}^{(i)}}{\mathbf{e}^T\boldsymbol{\pi}^{(i)}}$$

is the Perron vector for the associated Perron complement $\mathbf{P}_{ii}$. In what follows, the normalizing scalar

$$\xi_i \equiv \mathbf{e}^T\boldsymbol{\pi}^{(i)} > 0$$

will be called the $i$th *coupling factor*. Observe that the Perron vector for the larger matrix $\mathbf{A}$ can be written in terms of the Perron vectors of the smaller Perron complements as

$$\boldsymbol{\pi} = \begin{pmatrix} \xi_1\mathbf{p}_1 \\ \xi_2\mathbf{p}_2 \\ \vdots \\ \xi_k\mathbf{p}_k \end{pmatrix}. \tag{3.1}$$

This illustrates that—at least in theory—it is possible to *uncouple* the Perron eigenvector problem by using Perron complementation. If the matrix $\mathbf{A}$ is partitioned to $k$ levels so as to yield $k$ Perron complements—one $\mathbf{P}_{ii}$ for each diagonal block—then the Perron vectors $\mathbf{p}_i$ for the $\mathbf{P}_{ii}$'s can be determined independent of each other and then combined in order to construct the Perron vector $\boldsymbol{\pi}$ for the larger matrix $\mathbf{A}$. However, the expressions for the coupling factors $\xi_i$ have the form

$$\xi_i = \mathbf{e}^T\boldsymbol{\pi}^{(i)} = \sum_h \pi_h^{(i)}.$$

At first glance, this might seem to place the issue in a hopeless circle, because prior knowledge of $\boldsymbol{\pi}$—in the form of the sums $\sum_h \pi_h^{(i)}$—is necessary in order to reconstruct $\boldsymbol{\pi}$ from the $\mathbf{p}_i$'s. Fortunately, this is not the case. The following result shows that the coupling factors $\xi_i$ can very easily be determined without prior knowledge of the $\boldsymbol{\pi}^{(i)}$'s. This is a key feature in the uncoupling-coupling technique.

DEFINITION 3.2. Let $\mathbf{A}$ be a nonnegative irreducible matrix with spectral radius $\rho$. Suppose that $\mathbf{A}$ is partitioned as

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1k} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{k1} & \mathbf{A}_{k2} & \cdots & \mathbf{A}_{kk} \end{pmatrix}$$

with square diagonal blocks, and let $\mathbf{p}_i$ be the Perron vector for the Perron complement

$$\mathbf{P}_{ii} = \mathbf{A}_{ii} + \mathbf{A}_{i*}(\rho\mathbf{I} - \mathbf{A}_i)^{-1}\mathbf{A}_{*i}.$$

The *coupling matrix* associated with $\mathbf{A}$ is defined to be the $k \times k$ matrix $\mathbf{C} = [c_{ij}]$ whose entries are given by[2]

$$c_{ij} \equiv \mathbf{e}^T\mathbf{A}_{ij}\mathbf{p}_j.$$

[2]Throughout, the size of the vector $\mathbf{e}^T = (1\ 1\ \cdots\ 1)$ will always be defined by the context in which it appears.

The reason for this definition will immediately become apparent. The utility of the coupling matrix is realized in the following theorem, which demonstrates how the Perron vectors of the individual Perron complements can be easily coupled together by means of the coupling matrix $\mathbf{C}$ in order to construct the Perron vector for $\mathbf{A}$.

THEOREM 3.1. *For a nonnegative irreducible matrix*

$$\mathbf{A}=\begin{pmatrix}\mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1k}\\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2k}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{A}_{k1} & \mathbf{A}_{k2} & \cdots & \mathbf{A}_{kk}\end{pmatrix}$$

*with spectral radius* $\rho$ *and conformably partitioned Perron vector*

$$\boldsymbol{\pi}=\begin{pmatrix}\boldsymbol{\pi}^{(1)}\\ \boldsymbol{\pi}^{(2)}\\ \vdots\\ \boldsymbol{\pi}^{(k)}\end{pmatrix},$$

*the associated* $k\times k$ *coupling matrix* $\mathbf{C}$ *is also a nonnegative irreducible matrix whose spectral radius is again given by* $\rho$. *Furthermore, the Perron vector for* $\mathbf{C}$ *is precisely the vector*

$$\boldsymbol{\xi}=\begin{pmatrix}\xi_1\\ \xi_2\\ \vdots\\ \xi_k\end{pmatrix}$$

*whose components are the positive coupling factors defined earlier as*

$$\xi_i=\mathbf{e}^T\boldsymbol{\pi}^{(i)}.$$

*The vector* $\boldsymbol{\xi}$ *is hereafter referred to as the* coupling vector *associated with the partitioned matrix* $\mathbf{A}$.

*Proof.* It is clear from the definition of the coupling matrix that $\mathbf{C} \geqslant \mathbf{0}$. To see that $\mathbf{C}$ is irreducible, note that because

$$\mathbf{e}^T > \mathbf{0}, \qquad \mathbf{A}_{ij} \geqslant \mathbf{0}, \quad \text{and} \quad \mathbf{p}_j > 0,$$

it must be the case that

$$c_{ij} = 0 \quad \text{if and only if} \quad \mathbf{A}_{ij} = \mathbf{0}.$$

Since $\mathbf{A}$ is irreducible, the preceding statement implies that $\mathbf{C}$ must also be irreducible—otherwise, if $\mathbf{C}$ could be permuted to a block triangular form, then so could $\mathbf{A}$. To show that the coupling vector

$$\boldsymbol{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_k \end{pmatrix}$$

is the Perron vector for $\mathbf{C}$, use the fact

$$\mathbf{p}_j = \frac{\boldsymbol{\pi}^{(j)}}{\xi_j}$$

and compute the $i$th component of the product $\mathbf{C}\boldsymbol{\xi}$ to be

$$\begin{aligned}(\mathbf{C}\boldsymbol{\xi})_i = \sum_{j=1}^{k} c_{ij}\xi_j &= \sum_{j=1}^{k} \mathbf{e}^T\mathbf{A}_{ij}\mathbf{p}_j\xi_j \\ &= \mathbf{e}^T \sum_{j=1}^{k} \mathbf{A}_{ij}\boldsymbol{\pi}^{(j)} = \mathbf{e}^T\rho\boldsymbol{\pi}^{(i)} = \rho\xi_i.\end{aligned}$$

Thus $\mathbf{C}\boldsymbol{\xi} = \rho\boldsymbol{\xi}$, so that $\rho$ is a positive eigenvalue for $\mathbf{C}$ associated with a positive eigenvector $\boldsymbol{\xi}$. Since $\mathbf{C}$ is irreducible, $\rho$ must therefore be the spectral radius of $\mathbf{C}$. Because $\boldsymbol{\pi}$ is a normalized vector, it follows that $\boldsymbol{\xi}$ is a *normalized* positive eigenvector for $\mathbf{C}$ by observing

$$\sum_{i=1}^{k} \xi_i = \sum_{i=1}^{k} \mathbf{e}^T \boldsymbol{\pi}^{(i)} = \begin{pmatrix} \mathbf{e}^T & \mathbf{e}^T & \cdots & \mathbf{e}^T \end{pmatrix} \begin{pmatrix} \boldsymbol{\pi}^{(1)} \\ \boldsymbol{\pi}^{(2)} \\ \vdots \\ \boldsymbol{\pi}^{(k)} \end{pmatrix} = \mathbf{e}^T \boldsymbol{\pi} = 1.$$

■

By combining the observation of (3.1) with Theorem 3.1, we arrive at the major conclusion of this paper—which is summarized below as a formal statement.

THEOREM 3.2 (The coupling theorem). *Let* $\mathbf{A}$ *be a nonnegative irreducible matrix with spectral radius* $\rho$, *and suppose that* $\mathbf{A}$ *has a k-level partition*

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1k} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{k1} & \mathbf{A}_{k2} & \cdots & \mathbf{A}_{kk} \end{pmatrix}$$

*with square diagonal blocks. If* $\mathbf{p}_i$ *is the Perron vector for the Perron complement*

$$\mathbf{P}_{ii} = \mathbf{A}_{ii} + \mathbf{A}_{i*}(\rho \mathbf{I} - \mathbf{A}_i)^{-1} \mathbf{A}_{*i}$$

*and if*

$$\boldsymbol{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_k \end{pmatrix}$$

*is the Perron vector for the* $k \times k$ *coupling matrix* $\mathbf{C}$ *in which*

$$c_{ij} = \mathbf{e}^T \mathbf{A}_{ij} \mathbf{p}_j,$$

*then*

$$\boldsymbol{\pi} = \begin{pmatrix} \xi_1 \mathbf{p}_1 \\ \xi_2 \mathbf{p}_2 \\ \vdots \\ \xi_k \mathbf{p}_k \end{pmatrix}$$

*is the Perron vector for* $\mathbf{A}$. *Conversely, if the Perron vector for* $\mathbf{A}$ *is given in the form of a conformably partitioned vector*

$$\boldsymbol{\pi} = \begin{pmatrix} \boldsymbol{\pi}^{(1)} \\ \boldsymbol{\pi}^{(2)} \\ \vdots \\ \boldsymbol{\pi}^{(k)} \end{pmatrix},$$

*then* $\xi_i = \mathbf{e}^T \boldsymbol{\pi}^{(i)}$ *and* $\boldsymbol{\pi}^{(i)}/\xi_i$ *is the Perron vector for the Perron complement* $\mathbf{P}_{ii}$.

REMARK. The use of the coupling matrix is similar to techniques used in the theory of aggregation—e.g., see Courtois (1977) or Simon and Ando (1961) and the references contained therein. The coupling theorem given above might be viewed as an exact aggregation technique.

The case of a 2-level partition is of special interest. The following corollary shows that it is particularly easy to uncouple and couple the Perron eigenvector problem for these situations.

COROLLARY 3.1. *Let* $\mathbf{A}$ *be a nonnegative irreducible matrix with spectral radius* $\rho$. *If* $\mathbf{A}$ *is partitioned as*

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}$$

*where* $\mathbf{A}_{11}$ *and* $\mathbf{A}_{22}$ *are square, then the Perron vector for* $\mathbf{A}$ *is given by*

$$\boldsymbol{\pi} = \begin{pmatrix} \xi_1 \mathbf{p}_1 \\ \xi_2 \mathbf{p}_2 \end{pmatrix},$$

*where* $\mathbf{p}_1$ *and* $\mathbf{p}_2$ *are the Perron vectors for the Perron complements*

$$\mathbf{P}_{11} = \mathbf{A}_{11} + \mathbf{A}_{12}(\rho\mathbf{I} - \mathbf{A}_{22})^{-1}\mathbf{A}_{21} \quad \textit{and} \quad \mathbf{P}_{22} = \mathbf{A}_{22} + \mathbf{A}_{21}(\rho\mathbf{I} - \mathbf{A}_{11})^{-1}\mathbf{A}_{12},$$

*respectively, and where the coupling factors* $\xi_1$ *and* $\xi_2$ *are given by*

$$\xi_1 = \frac{\mathbf{e}^T\mathbf{A}_{12}\mathbf{p}_2}{\rho - \mathbf{e}^T\mathbf{A}_{11}\mathbf{p}_1 + \mathbf{e}^T\mathbf{A}_{12}\mathbf{p}_2} \quad \textit{and} \quad \xi_2 = 1 - \xi_1.$$

*Proof.* Simply verify that the given values of $\xi_1$ and $\xi_2$ are the two components of the Perron vector for the coupling matrix

$$C = \begin{pmatrix} \mathbf{e}^T\mathbf{A}_{11}\mathbf{p}_1 & \mathbf{e}^T\mathbf{A}_{12}\mathbf{p}_2 \\ \mathbf{e}^T\mathbf{A}_{21}\mathbf{p}_1 & \mathbf{e}^T\mathbf{A}_{22}\mathbf{p}_2 \end{pmatrix}.$$

■

There is always a balancing act to be performed when uncoupling the Perron eigenvector problem using Perron complementation. As $k$ increases and the partition of $\mathbf{A}$ becomes finer, the sizes of the Perron complements become smaller, thus making it easier to determine each of the Perron vectors $\mathbf{p}_i$. At the same time, however, the size of the coupling matrix $\mathbf{C}_{k\times k}$ becomes larger, thus making it more difficult to determine the coupling factors $\xi_i$. In the two extreme cases when either $k = m$ or $k = 1$, there is no uncoupling whatsoever, because $\mathbf{C} = \mathbf{A}$ when $k = m$ and $\mathbf{P}_{11} = \mathbf{A}$ when $k = 1$. Furthermore, as the size of a Perron complement becomes smaller, the order of the matrix inversion embedded in the Perron complement becomes larger. One must therefore choose the partition which best suits the needs of the underlying application. For example, if computation utilizing a particular parallel architecture is the goal, then the specific nature of the hardware and associated software may dictate the partitioning strategy.

Rather than performing a single uncoupling-coupling operation to a high level partition of $\mathbf{A}$, an alternative is to execute a divide-and-conquer procedure using only 2-level partitions at each stage. Starting with a nonnegative irreducible matrix $\mathbf{A}$ (with spectral radius $\rho$) of size $m \times m$, partition $\mathbf{A}$ roughly in half as

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}$$

to produce two Perron complements, $\mathbf{P}_{11}$ and $\mathbf{P}_{22}$, which are each nonnegative irreducible matrices (each having spectral radius $\rho$) of order approximately $m/2$. The Perron complements $\mathbf{P}_{11}$ and $\mathbf{P}_{22}$ may in turn be partitioned roughly in half to produce four Perron complements—say $(\mathbf{P}_{11})_{11}$, $(\mathbf{P}_{11})_{22}$, $(\mathbf{P}_{22})_{11}$, and $(\mathbf{P}_{22})_{22}$—each of which is of order approximately $m/4$ (again, each has spectral radius $\rho$). This process can continue until all Perron complements are sufficiently small in size so as to easily yield a Perron vector. The small Perron vectors are then very easy to successively couple—according to the rule given in Corollary 3.1—until the Perron vector $\pi$ for the original matrix is produced. At each stage, all Perron complements have spectral radius $\rho$, so that no eigenvalue computation past the initial determination of $\rho$ is necessary.

For example, consider the following $8\times 8$ nonnegative irreducible matrix whose spectral radius is $\rho \approx 33.2418$:

$$A = \left(\begin{array}{cccc|cccc} 8 & 6 & 3 & 5 & 7 & 0 & 7 & 1 \\ 0 & 7 & 3 & 8 & 5 & 6 & 4 & 1 \\ 1 & 2 & 6 & 1 & 3 & 8 & 8 & 7 \\ 2 & 8 & 4 & 0 & 7 & 7 & 8 & 2 \\ \hline 2 & 4 & 6 & 2 & 5 & 7 & 6 & 5 \\ 4 & 1 & 0 & 4 & 8 & 4 & 8 & 2 \\ 3 & 1 & 6 & 6 & 4 & 5 & 5 & 0 \\ 0 & 1 & 1 & 6 & 7 & 0 & 3 & 4 \end{array}\right).$$

For the indicated partition, the Perron complements of $\mathbf{A}$ are given by[3]

$$\mathbf{P}_{11} = \left(\begin{array}{cc|cc} 10.51 & 8.136 & 7.639 & 9.309 \\ 3.058 & 9.189 & 7.175 & 12.74 \\ \hline 5.231 & 4.915 & 11.91 & 8.862 \\ 6.401 & 11.16 & 10.38 & 7.113 \end{array}\right) \quad \text{and}$$

$$\mathbf{P}_{22} = \left(\begin{array}{cc|cc} 8.738 & 11.66 & 11.03 & 7.575 \\ 11.37 & 6.592 & 11.72 & 3.106 \\ \hline 8.567 & 10.15 & 11.11 & 2.880 \\ 9.455 & 2.842 & 5.969 & 5.141 \end{array}\right),$$

[3]The arithmetic indicated in this example is not exact. Numbers have been rounded to four significant digits.

with coupling vector

$$\xi = \begin{pmatrix} .5469 \\ .4531 \end{pmatrix}.$$

Now, the two Perron complements of $\mathbf{P}_{11}$ are

$$(\mathbf{P}_{11})_{11} = \begin{pmatrix} 17.50 & 17.65 \\ 11.27 & 20.61 \end{pmatrix} \quad \text{and} \quad (\mathbf{P}_{11})_{22} = \begin{pmatrix} 16.01 & 15.14 \\ 17.34 & 17.93 \end{pmatrix}$$

with coupling vector

$$\xi_1 = \begin{pmatrix} .5103 \\ .4897 \end{pmatrix},$$

while the two Perron complements of $\mathbf{P}_{22}$ are

$$(\mathbf{P}_{22})_{11} = \begin{pmatrix} 16.89 & 18.56 \\ 17.91 & 12.92 \end{pmatrix} \quad \text{and} \quad (\mathbf{P}_{22})_{22} = \begin{pmatrix} 26.25 & 9.965 \\ 16.05 & 10.35 \end{pmatrix}$$

with coupling vector

$$\xi_2 = \begin{pmatrix} .5595 \\ .4405 \end{pmatrix}.$$

The Perron vector for $(\mathbf{P}_{11})_{11}$ is given by $\begin{pmatrix} .5286 \\ .4714 \end{pmatrix}$, and the Perron vector for $(\mathbf{P}_{11})_{22}$ is $\begin{pmatrix} .4690 \\ .5310 \end{pmatrix}$, so that the Perron vector for $\mathbf{P}_{11}$ is

$$\mathbf{p}_1 = \begin{pmatrix} .5103 \times \begin{pmatrix} .5286 \\ .4714 \end{pmatrix} \\ .4897 \times \begin{pmatrix} .4690 \\ .5310 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} .2697 \\ .2406 \\ .2297 \\ .2600 \end{pmatrix}.$$

Similarly, the Perron vector for $(\mathbf{P}_{22})_{11}$ is given by $\begin{pmatrix} .5316 \\ .4684 \end{pmatrix}$ and the Perron vector for $(\mathbf{P}_{22})_{22}$ is $\begin{pmatrix} .5878 \\ .4122 \end{pmatrix}$, so that the Perron vector for $\mathbf{P}_{22}$ is

$$\mathbf{p}_2 = \begin{pmatrix} .5595 \times \begin{pmatrix} .5316 \\ .4684 \end{pmatrix} \\ .4405 \times \begin{pmatrix} .5878 \\ .4122 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} .2974 \\ .2621 \\ .2589 \\ .1816 \end{pmatrix}.$$

Therefore, the Perron vector for **A** must be

$$\boldsymbol{\pi} = \begin{pmatrix} \xi_1 \mathbf{p}_1 \\ \xi_2 \mathbf{p}_2 \end{pmatrix} = \begin{pmatrix} .5469 \times \begin{pmatrix} .2697 \\ .2406 \\ .2297 \\ .2600 \end{pmatrix} \\ .4531 \times \begin{pmatrix} .2974 \\ .2621 \\ .2589 \\ .1816 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} .1475 \\ .1316 \\ .1256 \\ .1422 \\ .1347 \\ .1188 \\ .1173 \\ .0823 \end{pmatrix}.$$

In addition to the divide-and-conquer process illustrated above, there are several other variations and hybrid techniques—e.g., iterative methods—which are possible. It is clear that the remarks of this section can serve as the basis for a fully parallel algorithm for computing the Perron vector for a nonnegative irreducible matrix. However, there are several substantial computational issues which must be addressed, and therefore a more detailed discussion concerning parallel implementations will be considered in separate papers.

## 4. PRIMITIVITY ISSUES

Primitivity is, of course, an important issue. Accordingly, it is worthwhile to make some observations concerning the degree to which primitivity—or lack of primitivity—in a partitioned matrix **A** is inherited by the smaller Perron complements $\mathbf{P}_{ii}$.

The first observation to make is that **A** being a primitive matrix is not sufficient to guarantee that all Perron complements are primitive. For example, the matrix

$$A = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \hline 1 & 0 & 0 \end{array}\right)$$

is irreducible and primitive because $\mathbf{A}^5 > 0$. However, for the indicated partition, the Perron complement

$$\mathbf{P}_{11} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is not primitive.

Nevertheless, it is true that if a particular diagonal block in **A** is primitive, then the corresponding Perron complement must also be primitive.

THEOREM 4.1. *Let* $\mathbf{A}$ *be a nonnegative irreducible matrix partitioned as*

$$\mathbf{A}=\begin{pmatrix}\mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1k}\\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2k}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{A}_{k1} & \mathbf{A}_{k2} & \cdots & \mathbf{A}_{kk}\end{pmatrix}$$

*in which all diagonal blocks are square. If a particular diagonal block* $\mathbf{A}_{ii}$ *is primitive, then the corresponding Perron complement* $\mathbf{P}_{ii}$ *must also be primitive.*

*Proof.* Since $\mathbf{A}_{ii} \geqslant 0$ and $\mathbf{A}_{i*}(\mathbf{I}-\mathbf{A}_i)^{-1}\mathbf{A}_{*i} \geqslant 0$, it follows that for each positive integer $n$,

$$\mathbf{P}_{ii}^n = \left[\mathbf{A}_{ii} + \mathbf{A}_{i*}(\mathbf{I}-\mathbf{A}_i)^{-1}\mathbf{A}_{*i}\right]^n = \mathbf{A}_{ii}^n + \mathbf{N},$$

where $\mathbf{N} \geqslant 0$. Therefore, $\mathbf{P}_{ii}^n > 0$ whenever $\mathbf{A}_{ii}^n > \mathbf{0}$. ■

The following theorem explains why all but a special class of Perron complements must be primitive.

THEOREM 4.2. *If* $\mathbf{A}_{ii}$ *has at least one nonzero diagonal entry, then the corresponding Perron complement* $\mathbf{P}_{ii}$ *must be primitive.*

*Proof.* If $\mathbf{A}_{ii}$ has at least one nonzero diagonal entry, then so does $\mathbf{P}_{ii}$. Theorems 2.1 and 2.2 guarantee that each $\mathbf{P}_{ii}$ is always nonnegative and irreducible. It is well known—see Berman and Plemmons (1979, p. 34)—that an irreducible nonnegative matrix with a positive trace must be primitive. Therefore, $\mathbf{P}_{ii}$ must be primitive. ■

The converse of Theorem 4.1 as well as the converse of Theorem 4.2 is false. The matrix

$$\mathbf{A}=\left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0\\ \hline 0 & \frac{1}{2} & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right) \tag{4.1}$$

is irreducible, but notice that neither $\mathbf{A}_{11}$, $\mathbf{A}_{22}$, nor $\mathbf{A}$ itself is primitive.

Nevertheless, the corresponding Perron complements

$$\mathbf{P}_{11}=\begin{pmatrix}0 & \frac{1}{2} & 1\\ 1 & 0 & 0\\ 0 & \frac{1}{2} & 0\end{pmatrix} \quad \text{and} \quad \mathbf{P}_{22}=\begin{pmatrix}0 & 0 & \frac{1}{2}\\ 1 & 0 & \frac{1}{2}\\ 0 & 1 & 0\end{pmatrix}$$

are each primitive.

This example indicates another important advantage which Perron complementation can provide. The matrix $\mathbf{A}$ in (4.1) is not primitive, and hence its Perron vector $\boldsymbol{\pi}$ *cannot* be computed by the simple iteration[4]

$$\mathbf{x}_{n+1}=\left(\frac{\mathbf{A}}{\rho}\right)x_n, \qquad \text{where } \mathbf{x}_0 \text{ is arbitrary.}$$

However, $\mathbf{P}_{11}$ and $\mathbf{P}_{22}$ are each primitive, and therefore each $\mathbf{P}_{ii}$ yields a Perron vector $\mathbf{p}_i$ by means of the straightforward iterations

$$\mathbf{p}_i^{(n+1)}=\left(\frac{\mathbf{P}_{ii}}{\rho}\right)\mathbf{p}_i^{(n)}, \qquad \text{where } \mathbf{p}_i^{(0)} \text{ is arbitrary.}$$

Take note of the fact that the two iterations represented here are completely independent of each other and consequently they can be implemented simultaneously. By using the coupling factors described in Corollary 3.1, it is easy to couple $\mathbf{p}_1$ with $\mathbf{p}_2$ in order to produce the Perron vector for the larger imprimitive matrix $\mathbf{A}$.

## 5. SUMMARY

For a nonnegative irreducible matrix which is partitioned as a matrix with square diagonal blocks

$$\mathbf{A}_{m\times m}=\begin{pmatrix}\mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1k}\\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2k}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{A}_{k1} & \mathbf{A}_{k2} & \cdots & \mathbf{A}_{kk}\end{pmatrix},$$

[4]The spectral radius $\rho$ is included here for the sake of generality. Notice that the matrix in (4.1) is column stochastic, so that $\rho=1$.

which has spectral radius $\rho$ and conformably partitioned Perron vector

$$\boldsymbol{\pi} = \begin{pmatrix} \boldsymbol{\pi}^{(1)} \\ \boldsymbol{\pi}^{(2)} \\ \vdots \\ \boldsymbol{\pi}^{(k)} \end{pmatrix},$$

the concept of the *Perron complement* of $\mathbf{A}_{ii}$ is introduced as

$$\mathbf{P}_{ii} = \mathbf{A}_{ii} + \mathbf{A}_{i*}(\rho\mathbf{I} - \mathbf{A}_i)^{-1}\mathbf{A}_{*i}, \qquad i = 1, 2, \ldots, k.$$

Perron complementation is shown to possess the following properties:

(1) Each $\mathbf{P}_{ii}$ is also a nonnegative and irreducible matrix.
(2) Each $\mathbf{P}_{ii}$ also has spectral radius $\rho$.
(3) The Perron vector $\mathbf{p}_i$ for $\mathbf{P}_{ii}$ is the normalized $i$th segment of $\boldsymbol{\pi}$. That is, $\mathbf{p}_i = \boldsymbol{\pi}^{(i)}/\xi_i$, where the normalizing factors (also referred to as the *coupling factors*) are $\xi_i = \mathbf{e}^T\boldsymbol{\pi}^{(i)}$.
(4) The vector

$$\boldsymbol{\xi}_{k\times 1} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_k \end{pmatrix}$$

is the Perron vector for the $k \times k$ *coupling matrix* $\mathbf{C}$—which also is a nonnegative irreducible matrix with spectral radius $\rho$—whose entries are defined to be

$$c_{ij} = \mathbf{e}^T\mathbf{A}_{ij}\mathbf{p}_j.$$

These results allow the Perron eigenvector problem to be completely uncoupled in the sense that the Perron vector $\mathbf{p}_i$ of each small Perron complement $\mathbf{P}_{ii}$ can be determined independently of the Perron vectors of the other Perron complements. By using the Perron vector $\boldsymbol{\xi}$ for the coupling matrix $\mathbf{C}$, the smaller Perron vectors $\mathbf{p}_i$ associated with the individual Perron complements can be coupled together to form the Perron vector $\pi$ for the original large matrix $\mathbf{A}$ by taking

$$\pi = \begin{pmatrix} \xi_1 \mathbf{p}_1 \\ \xi_2 \mathbf{p}_2 \\ \vdots \\ \xi_k \mathbf{p}_k \end{pmatrix}.$$

*This work represents a generalization of the results of Meyer (1987), in which the concept of stochastic complementation was introduced in order to uncouple finite Markov chain problems.*

## REFERENCES


Berman, A. and Plemmons, R. J. 1979. *Nonnegative Matrices in The Mathematical Sciences*, Academic, New York.

Courtois, P. J. 1977. *Decomposability: Queueing and Computer System Applications*, Academic, New York.

Gantmacher, F. R. 1960. *Matrix Theory*, Vol. II, Chelsea, New York.

Johnson, C. R. 1981. Row stochastic matrices similar to doubly stochastic matrices, *Linear and Multilinear Algebra* 10:113–130.

Horn, R. A. and Johnson, C. R. 1985. *Matrix Analysis*, Cambridge U.P. New York.

Meyer, C. D. 1987. *Stochastic Complementation, Uncoupling Markov Chains, and the Simon-Ando Theory of Nearly Reducible Systems*, NCSU Center Res. Sci. Comp. Tech. Report 10018701, To appear in Siam Rev.

Simon, H. A. and Ando, A. 1961. Aggregation of variables in dynamic systems, *Econometrica* 29(2):111–138.


*Received 14 December 1987; final manuscript accepted 9 March 1988*